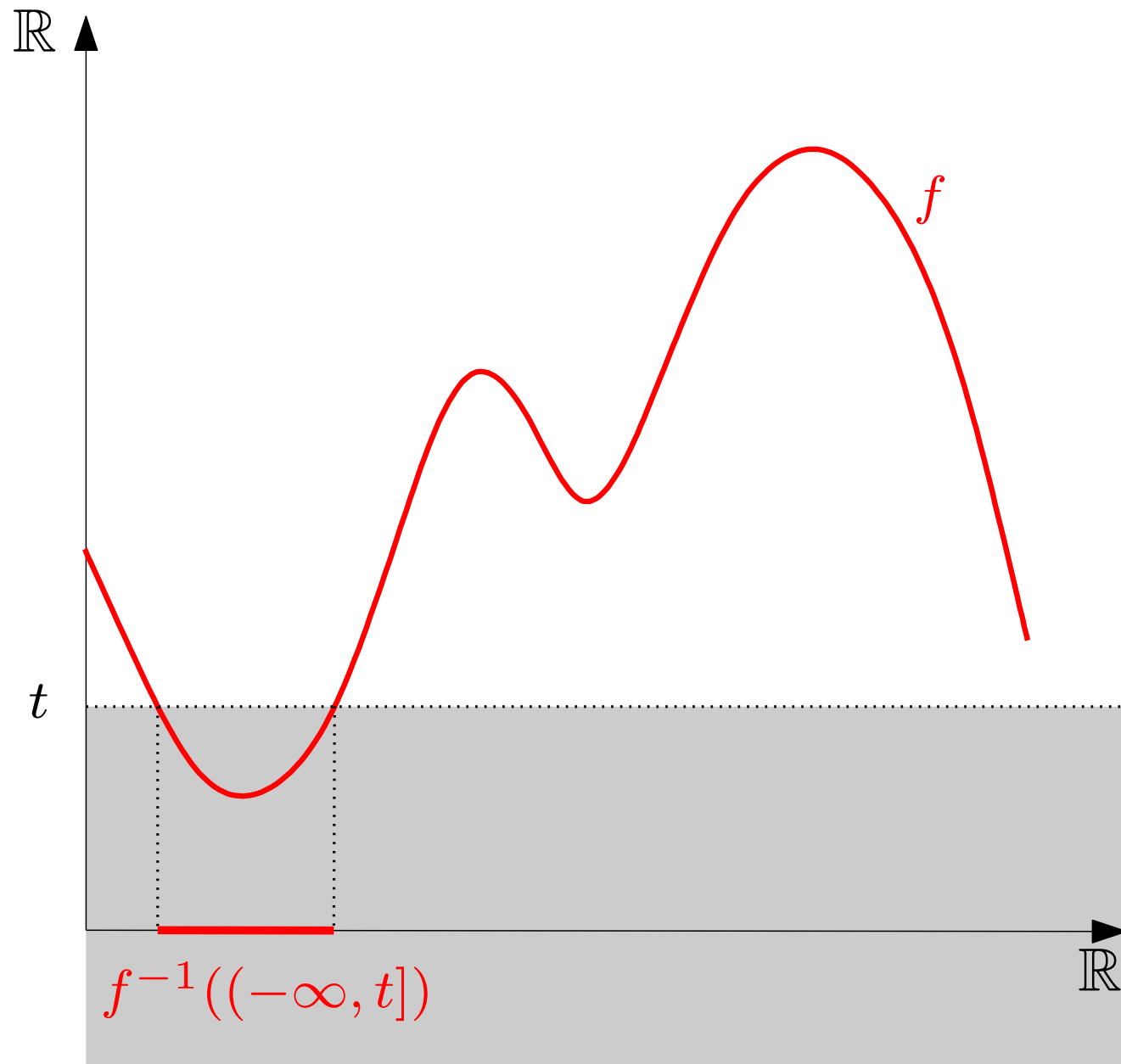


Persistent Homology

Smooth Real Function

Inside the black box:

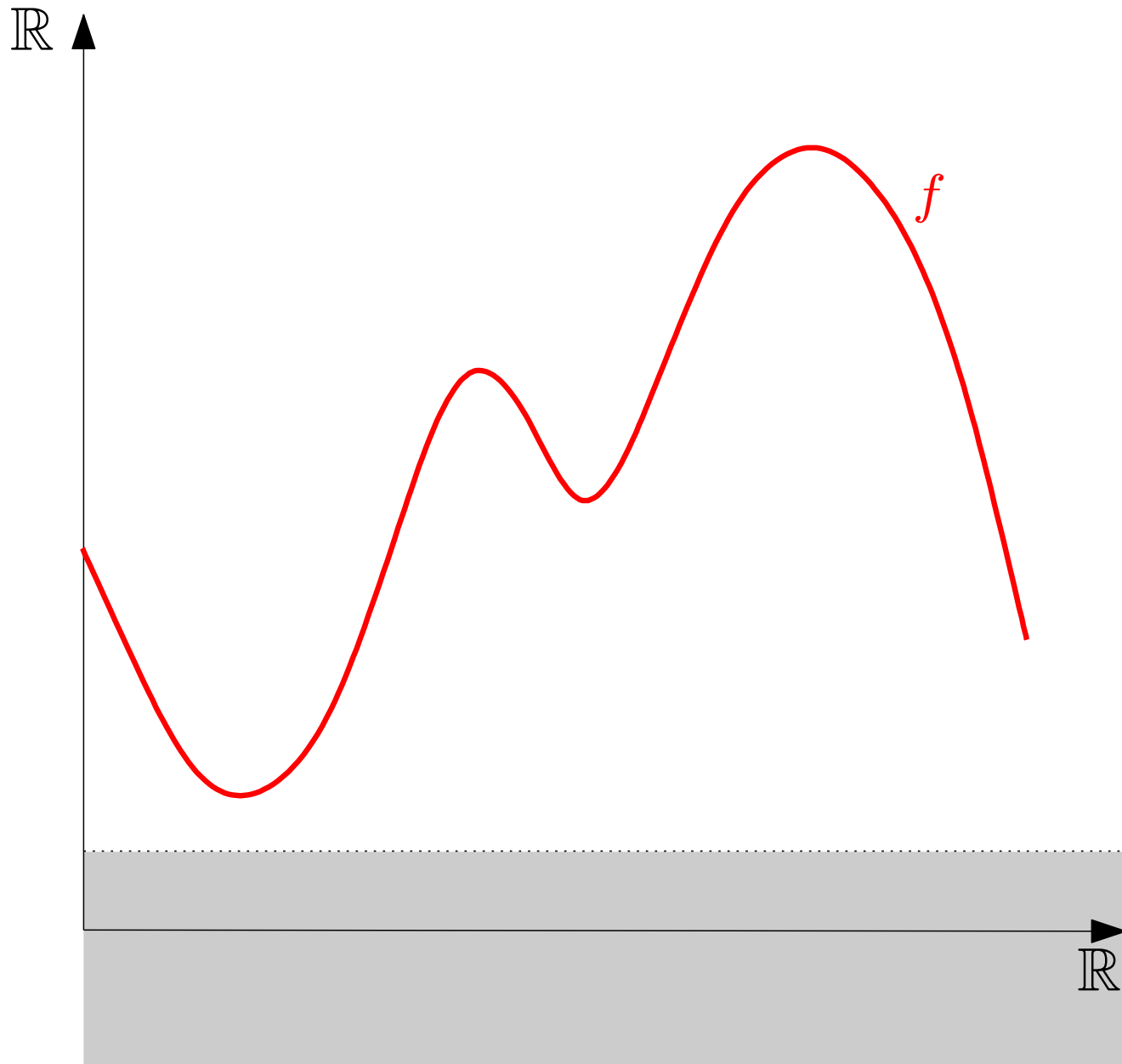
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family



Smooth Real Function

Inside the black box:

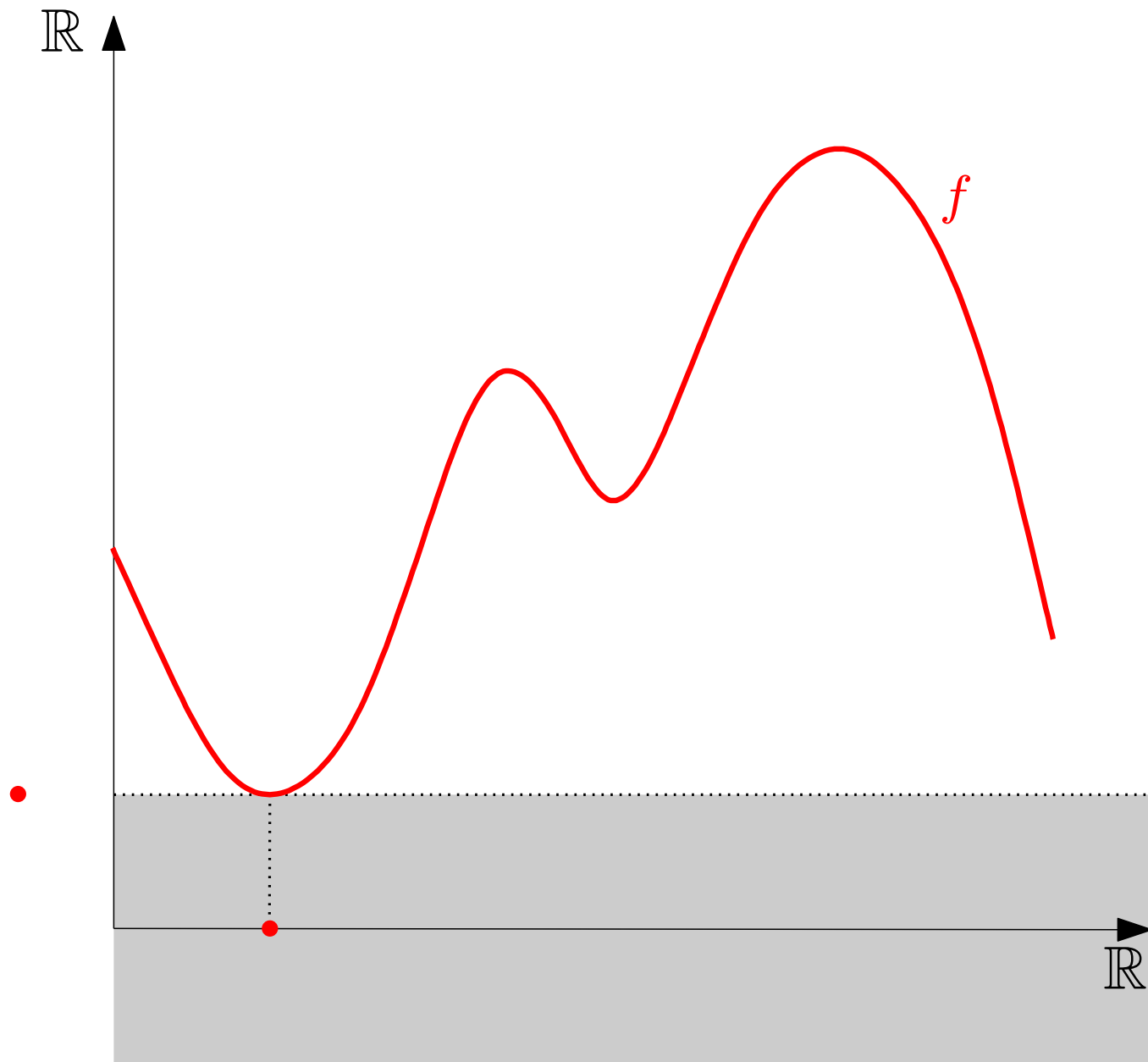
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Smooth Real Function

Inside the black box:

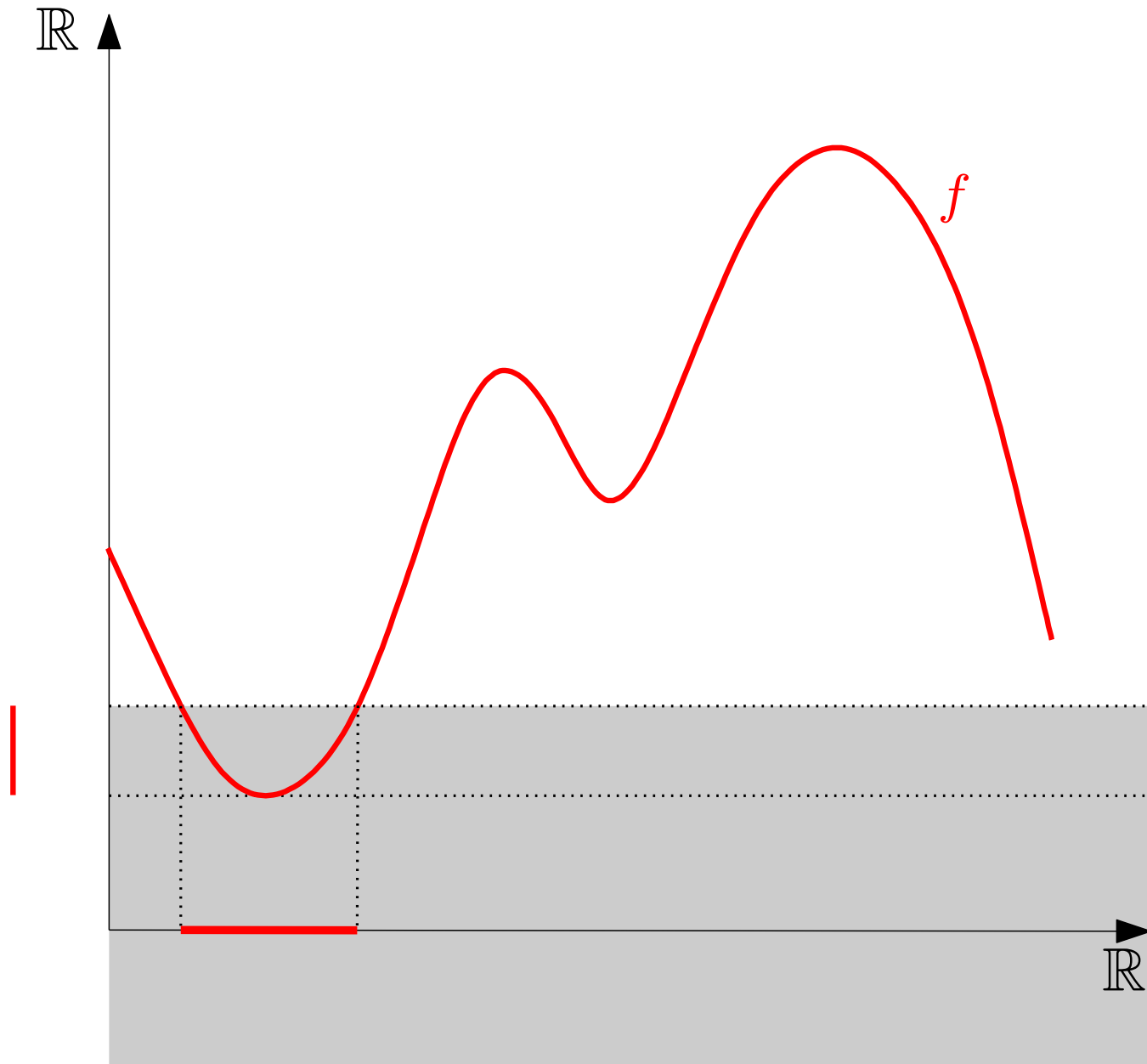
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Smooth Real Function

Inside the black box:

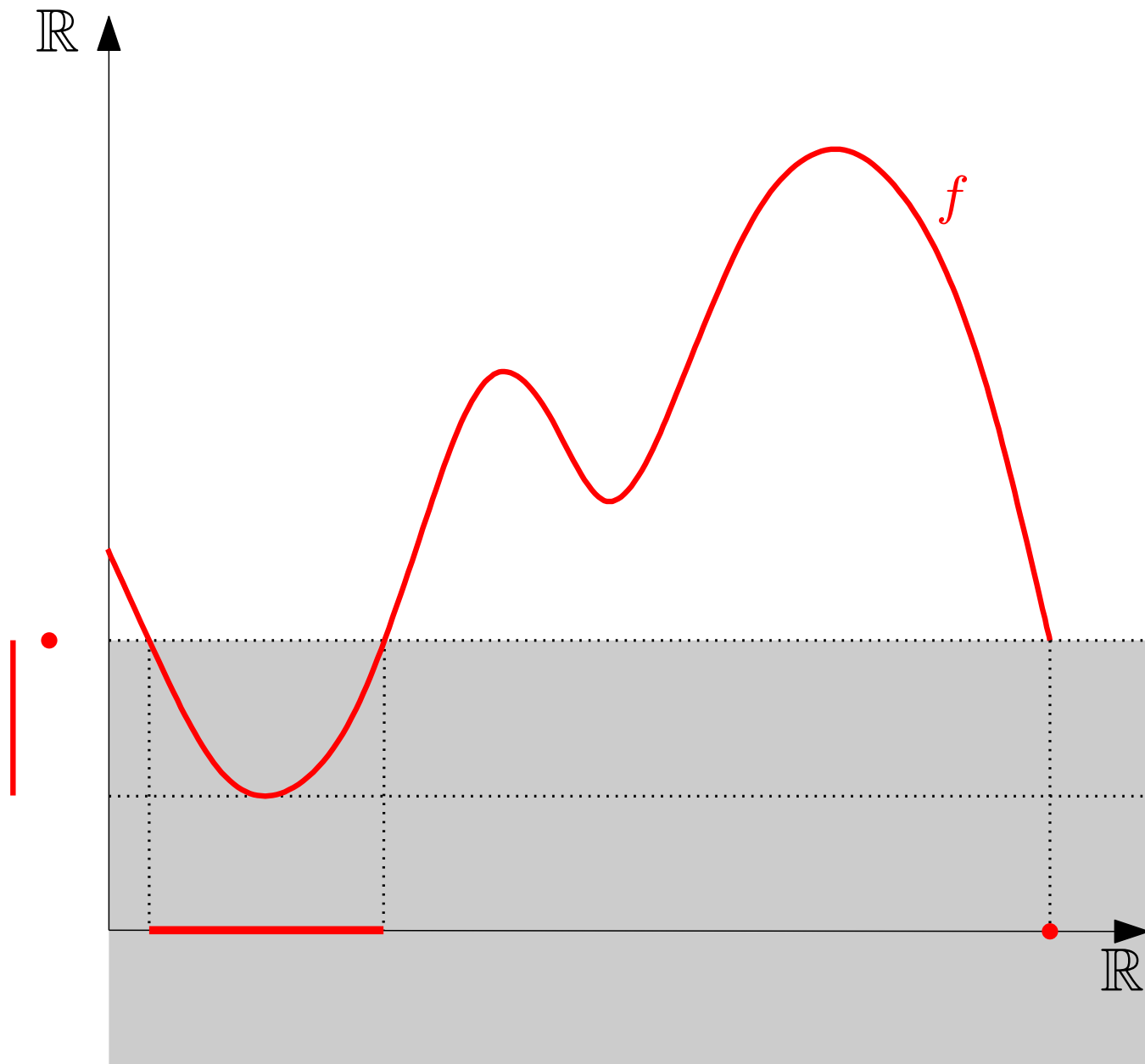
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Smooth Real Function

Inside the black box:

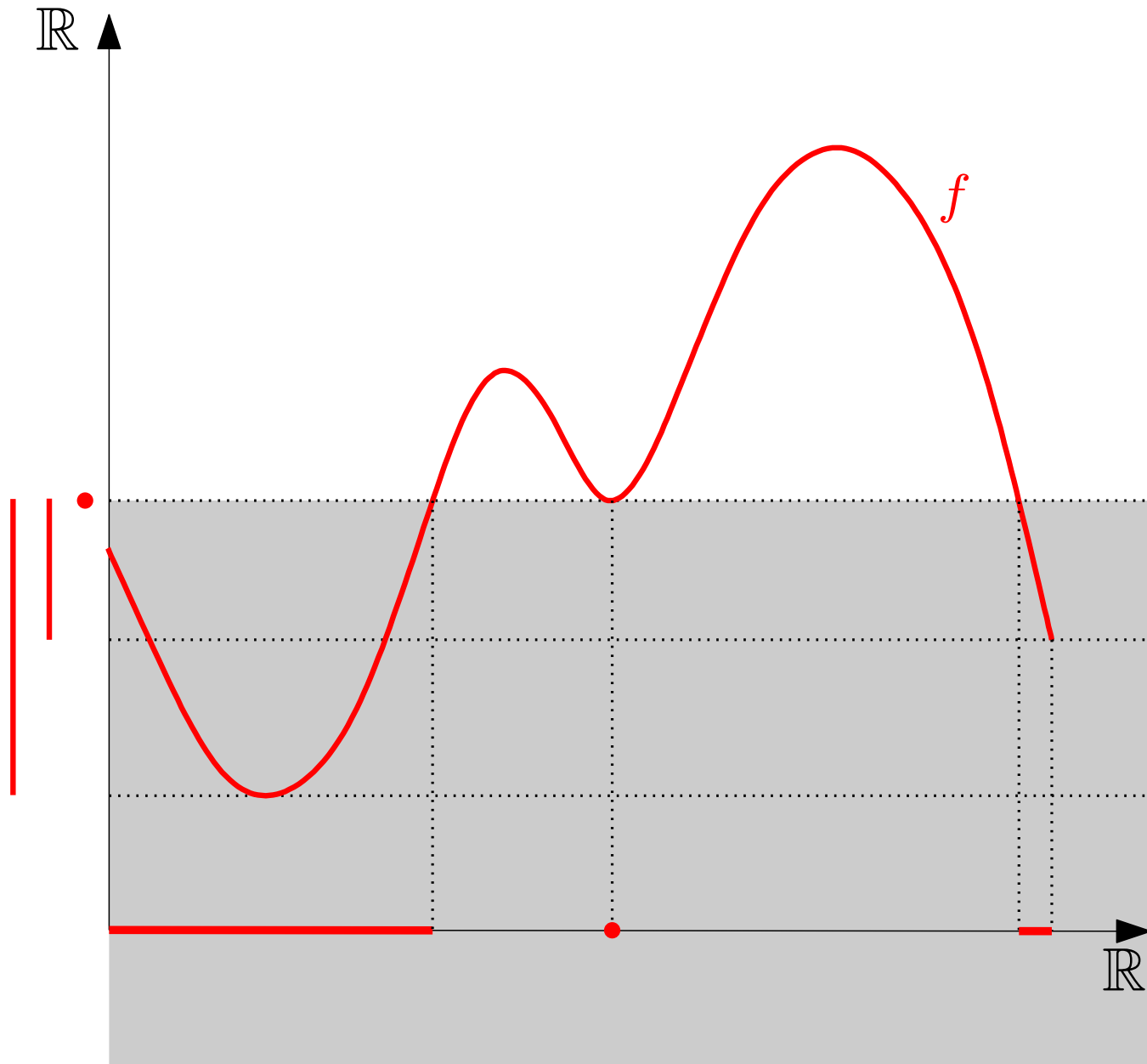
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Smooth Real Function

Inside the black box:

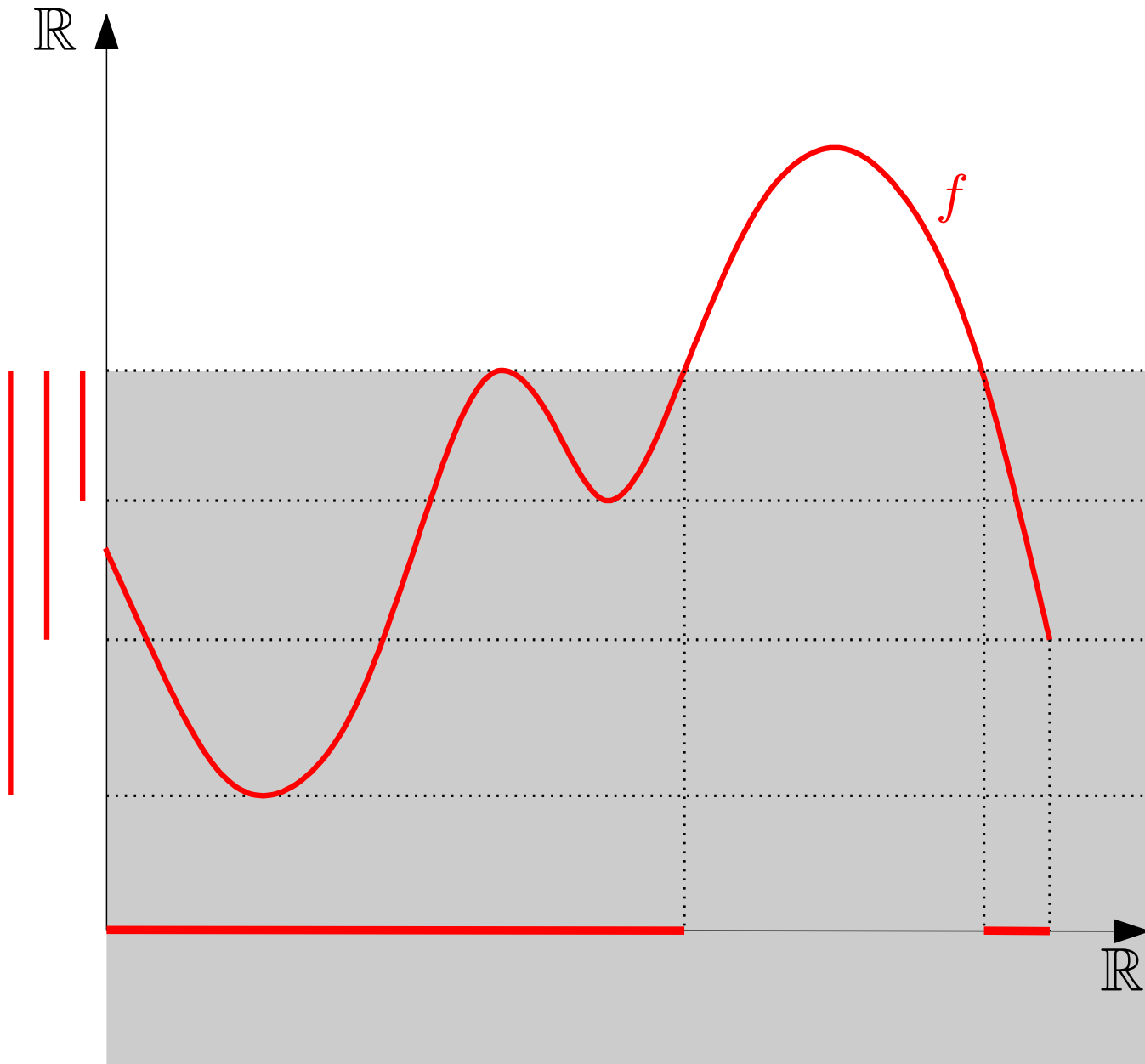
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Smooth Real Function

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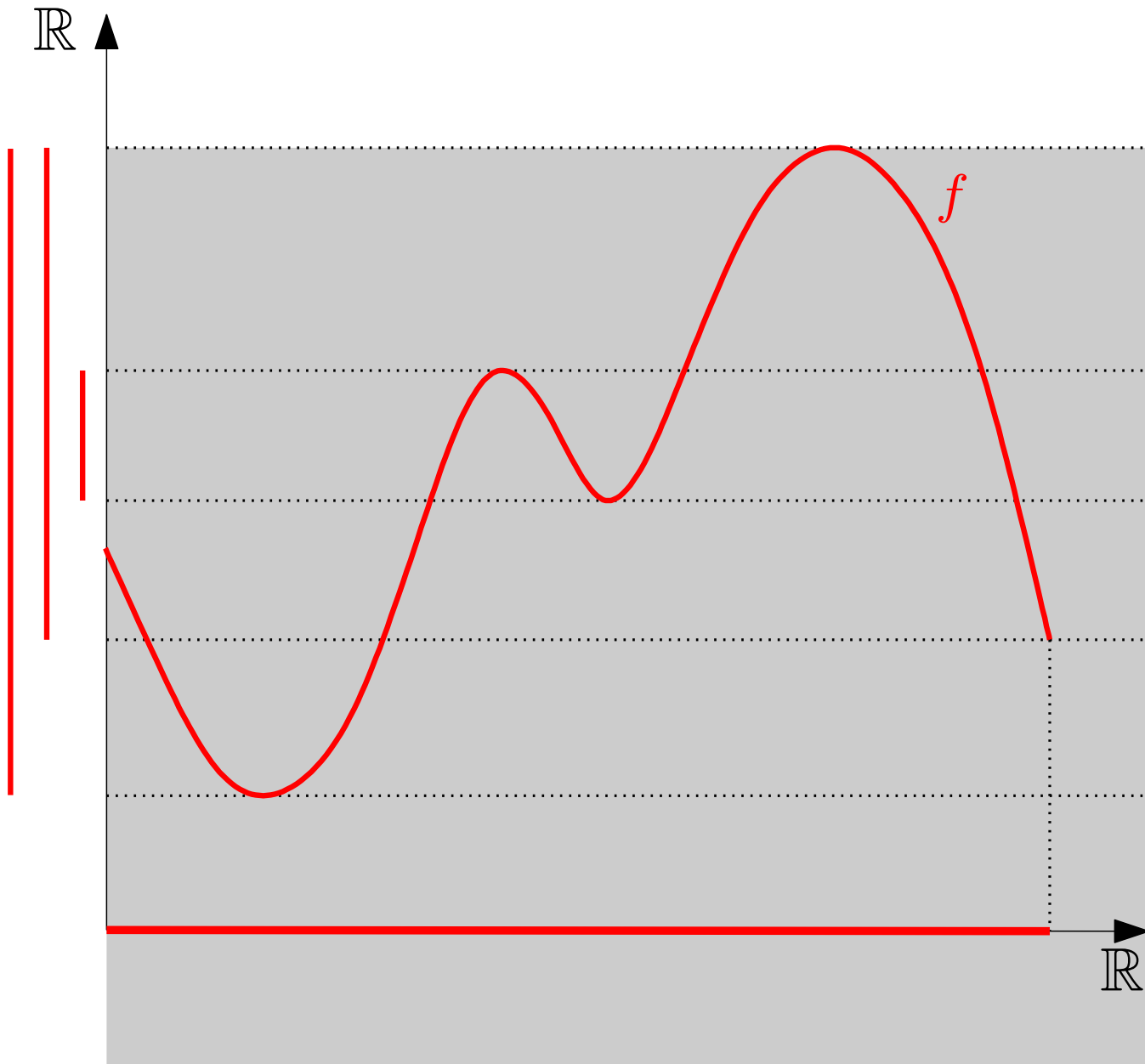
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Smooth Real Function

Inside the black box:

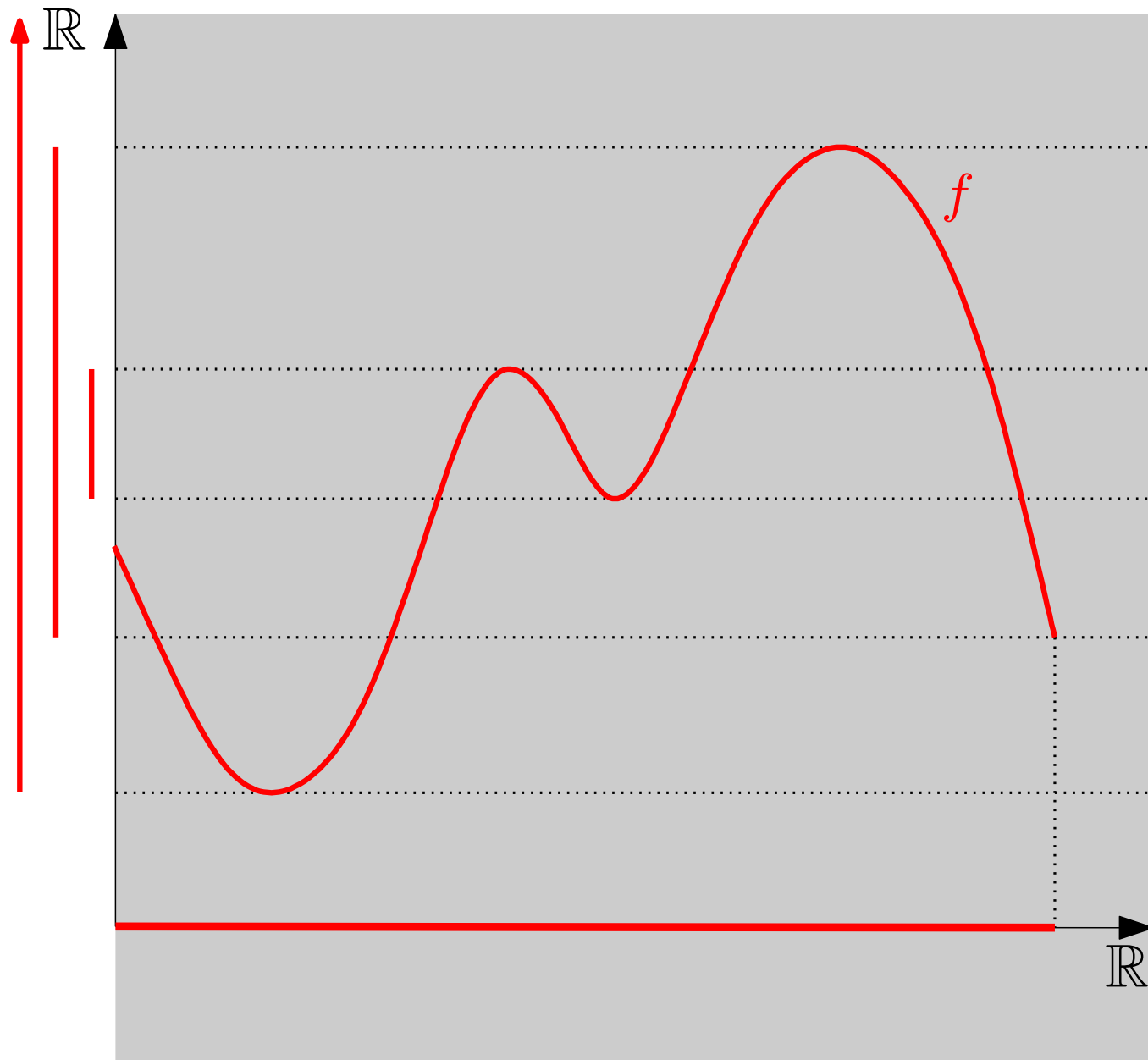
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Smooth Real Function

Inside the black box:

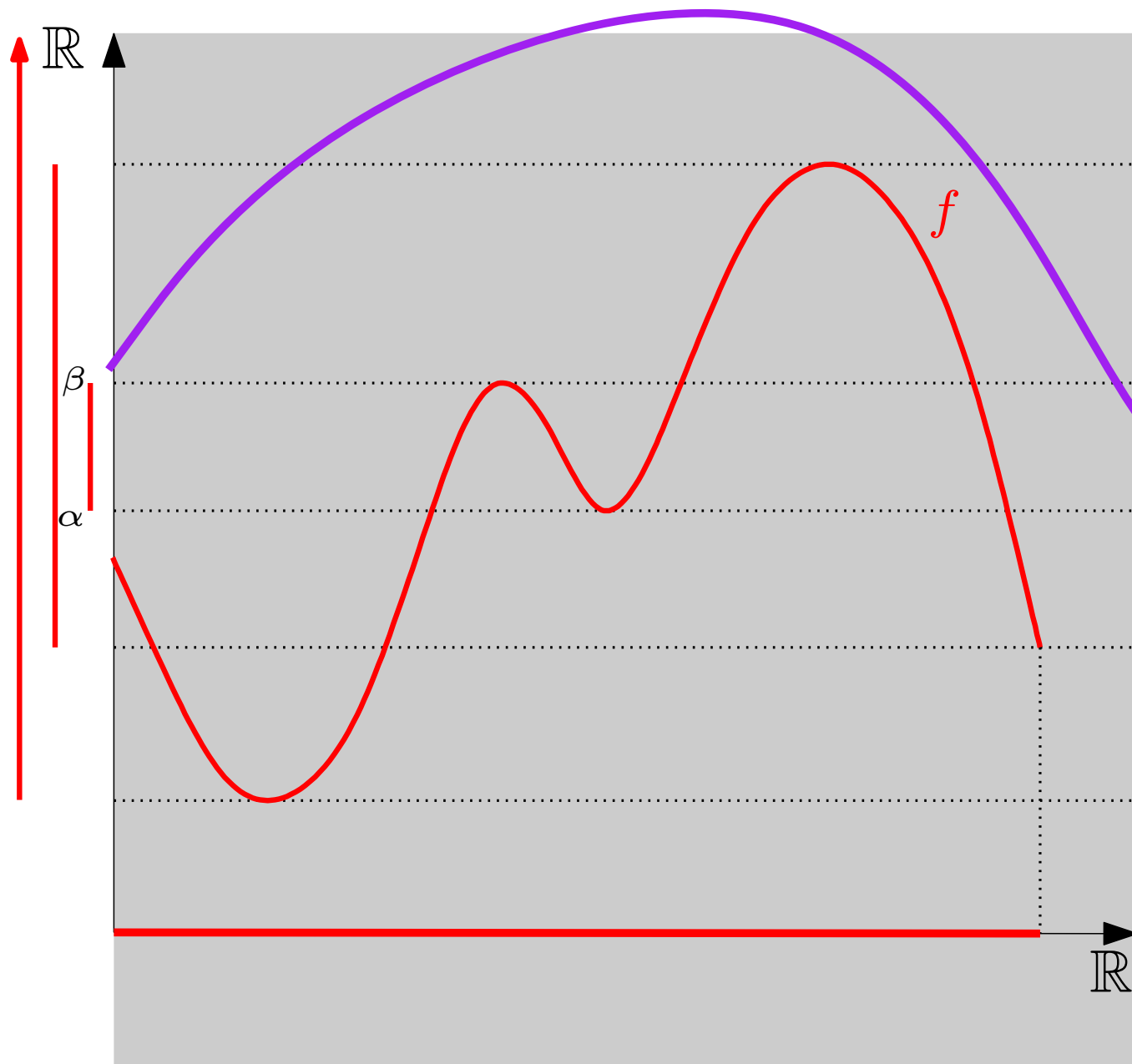
- Nested family (*filtration*) of sublevel-sets $f^{-1}((-\infty, t])$ for t ranging over $\mathbb{R}$
- Track the evolution of the topology (homology) throughout the family
- Finite set of intervals (barcode) encodes births/deaths of topological features



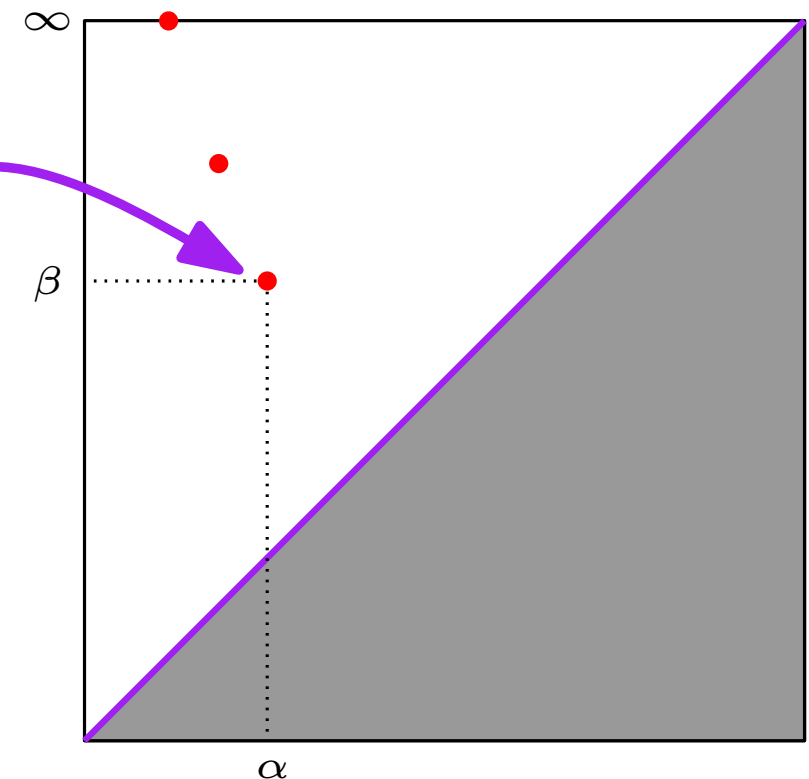
Smooth Real Function

Inside the black box:

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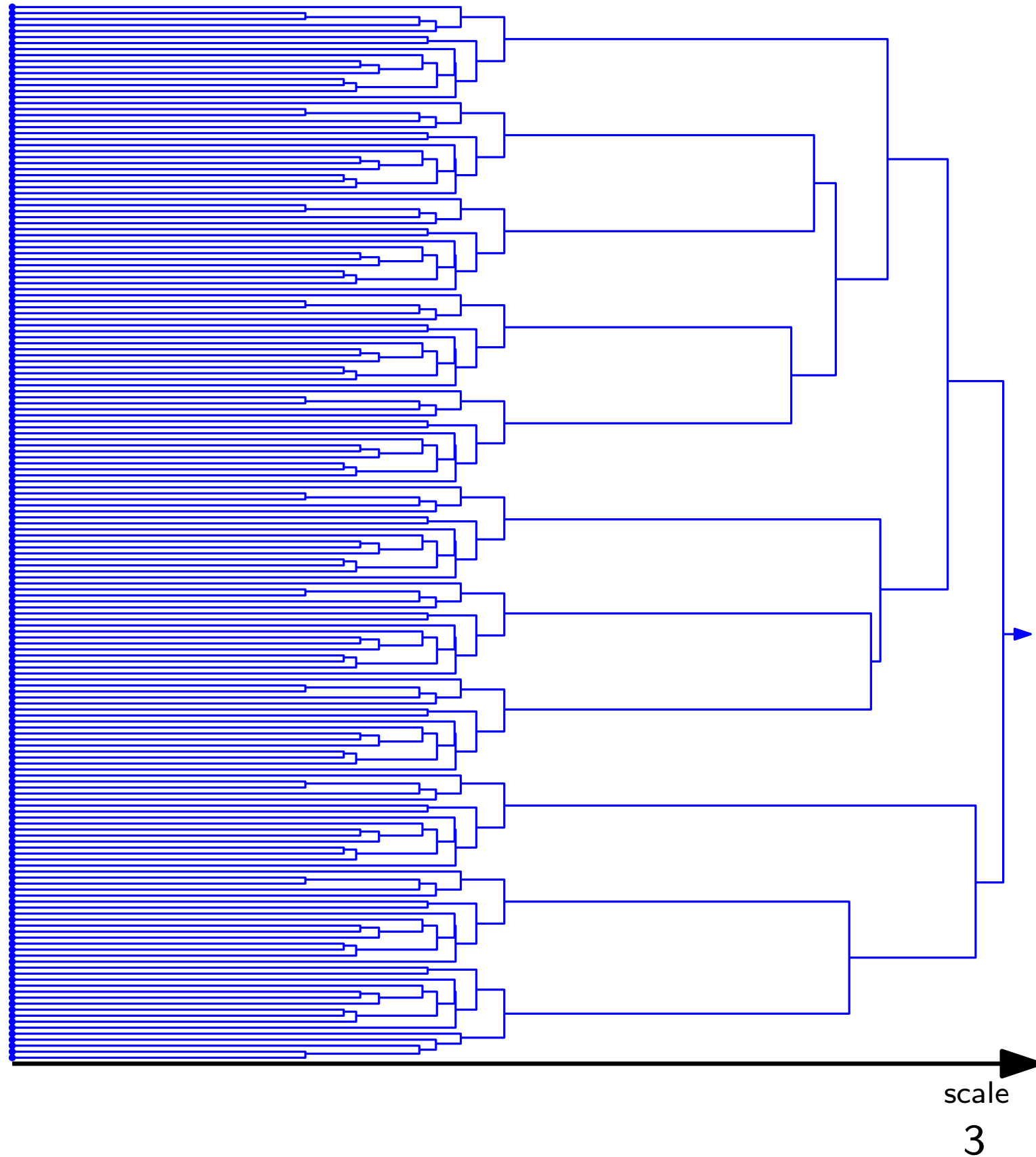
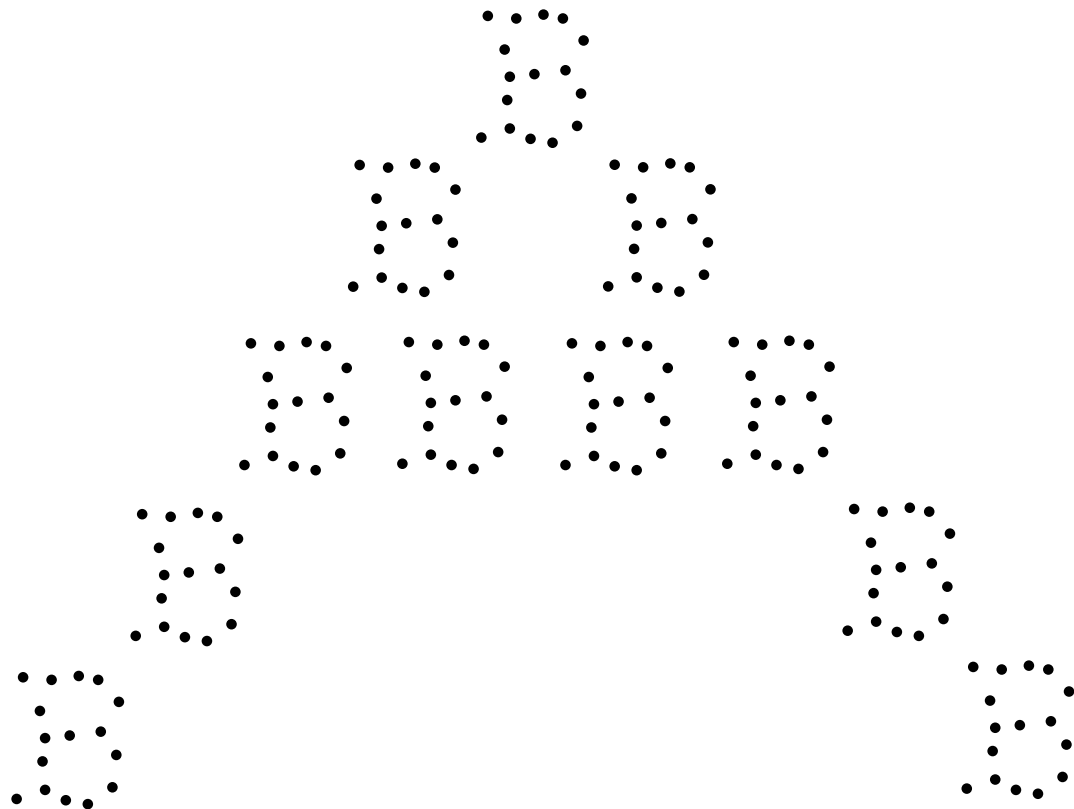


- Alternate representation as a (multi-) set of points in the plane (*diagram*).



Distance Function

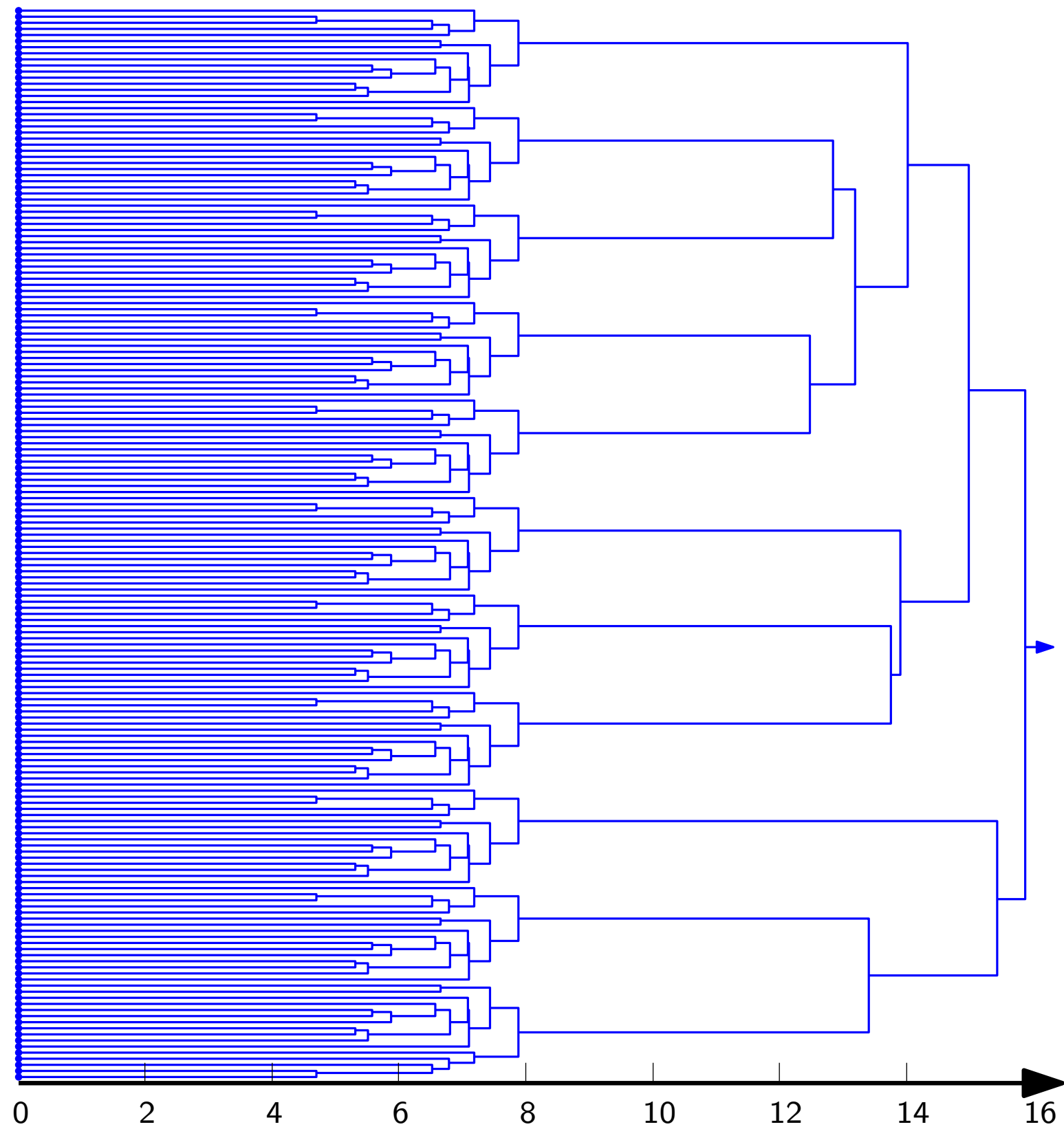
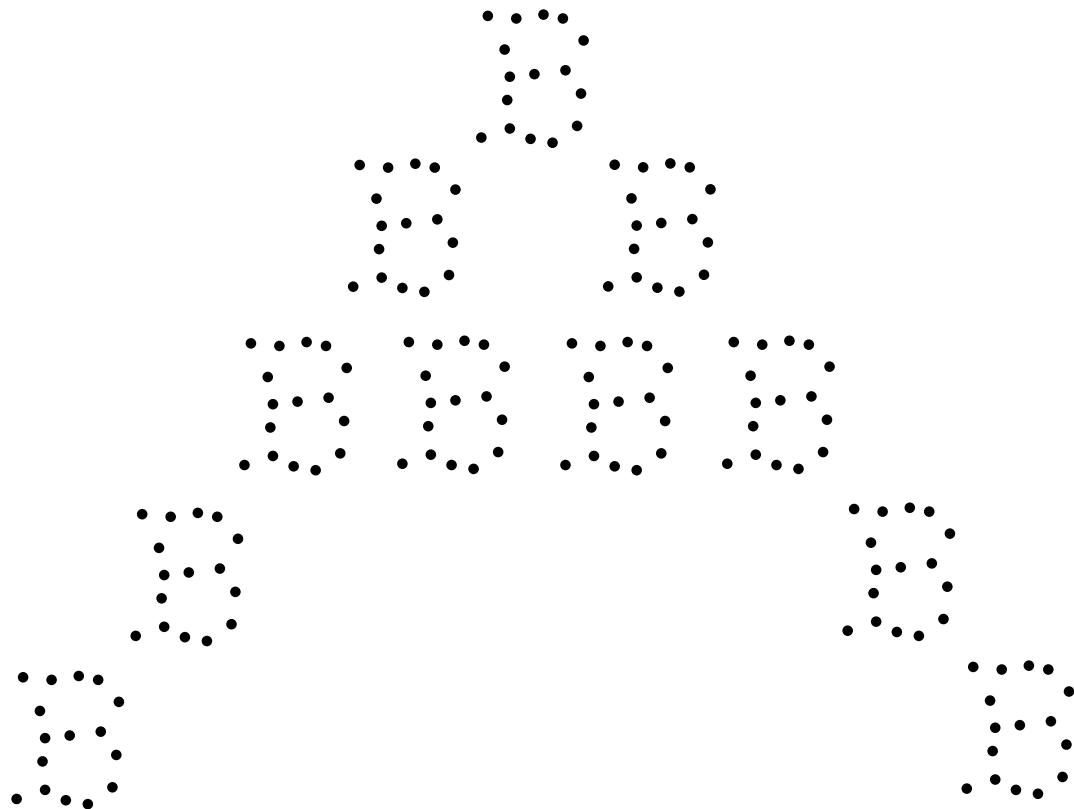
$$d_{\mathcal{P}} : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$



Distance Function

$$d_{\mathcal{P}} : \mathbb{R}^2 \rightarrow \mathbb{R}$$

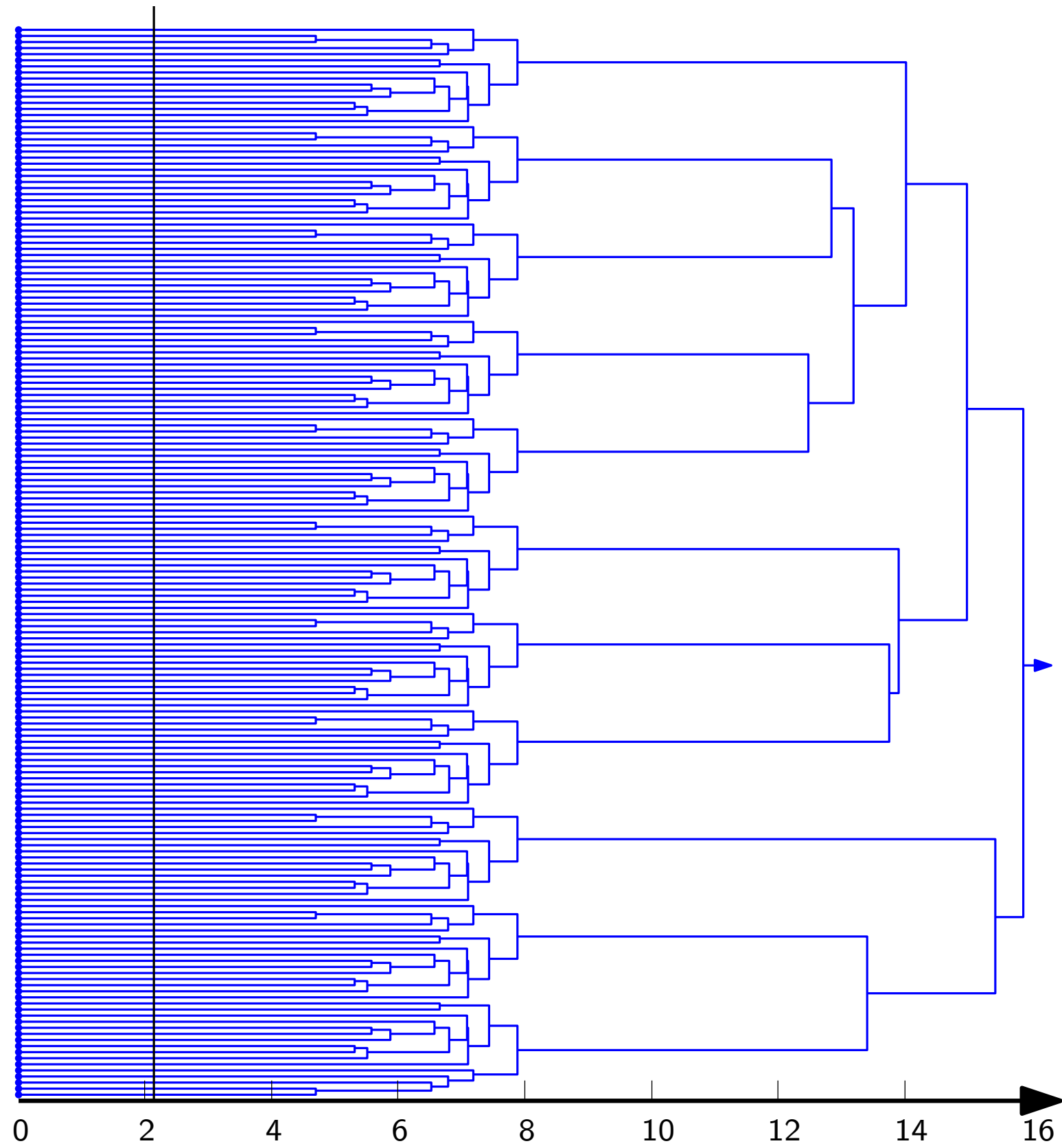
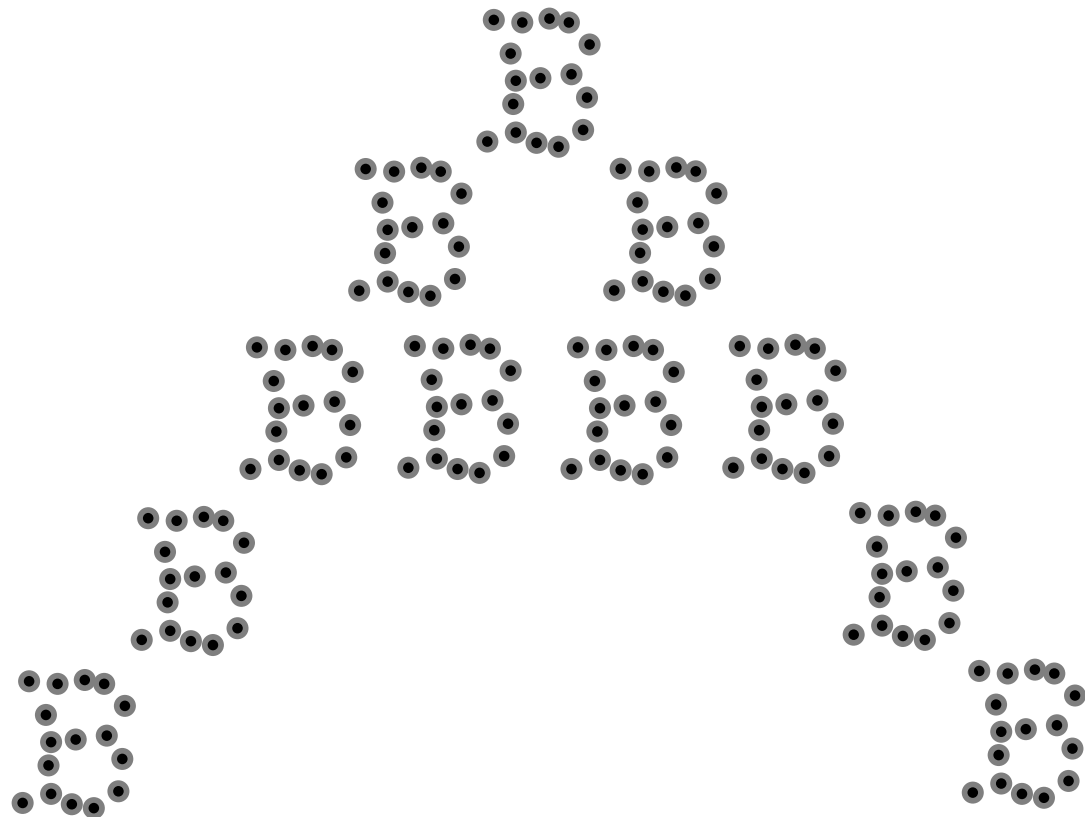
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Distance Function

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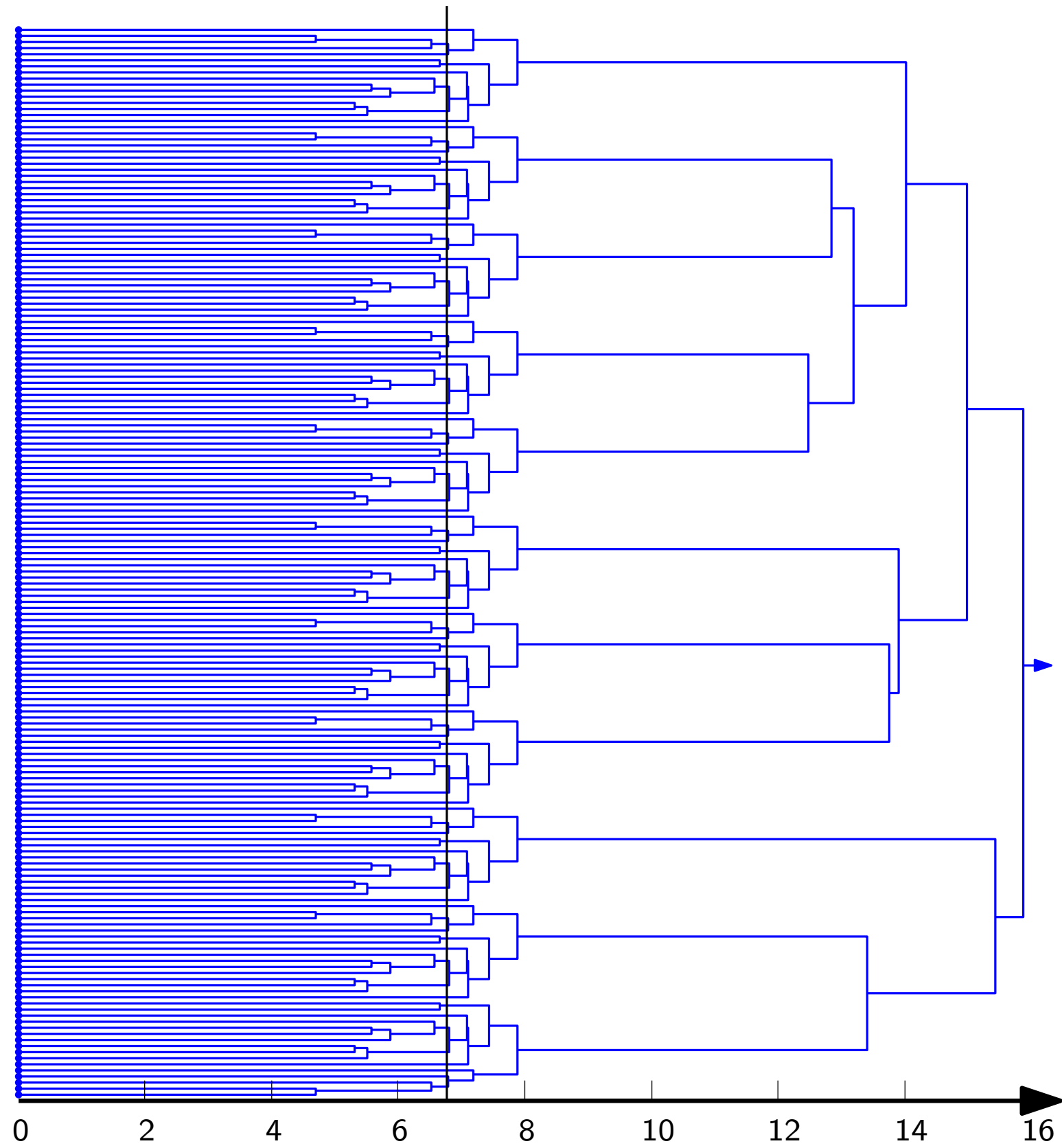
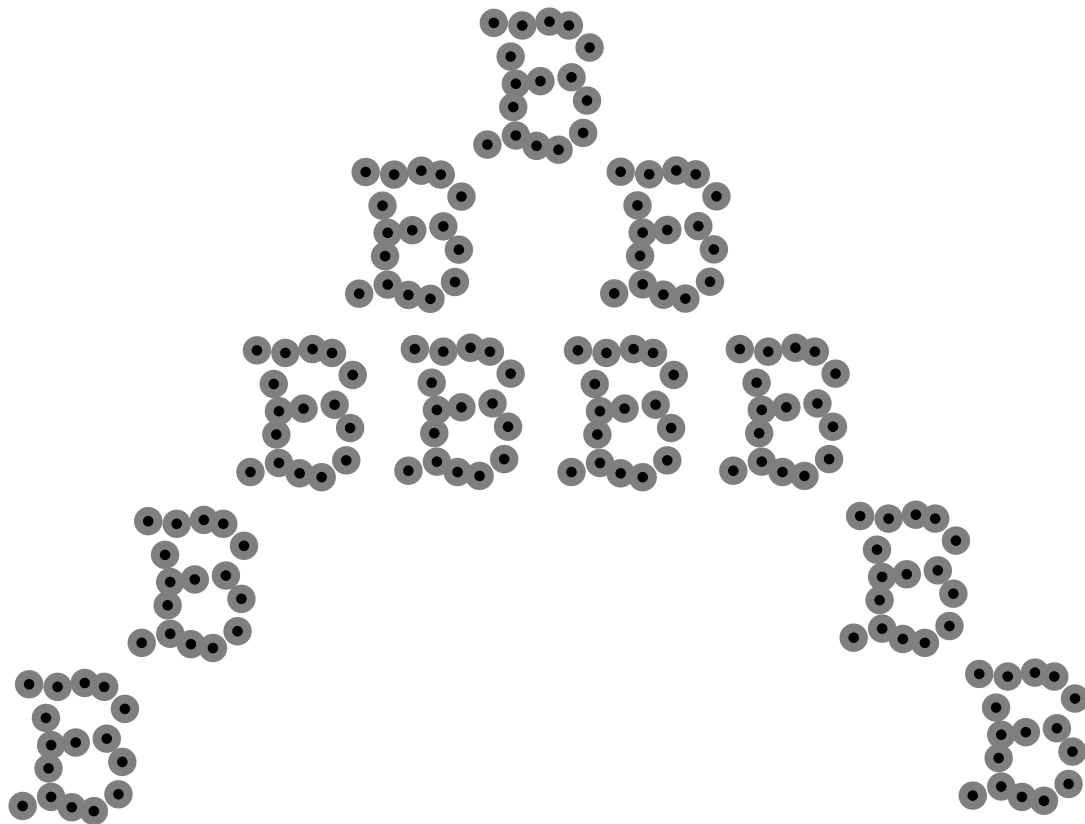
$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$



Distance Function

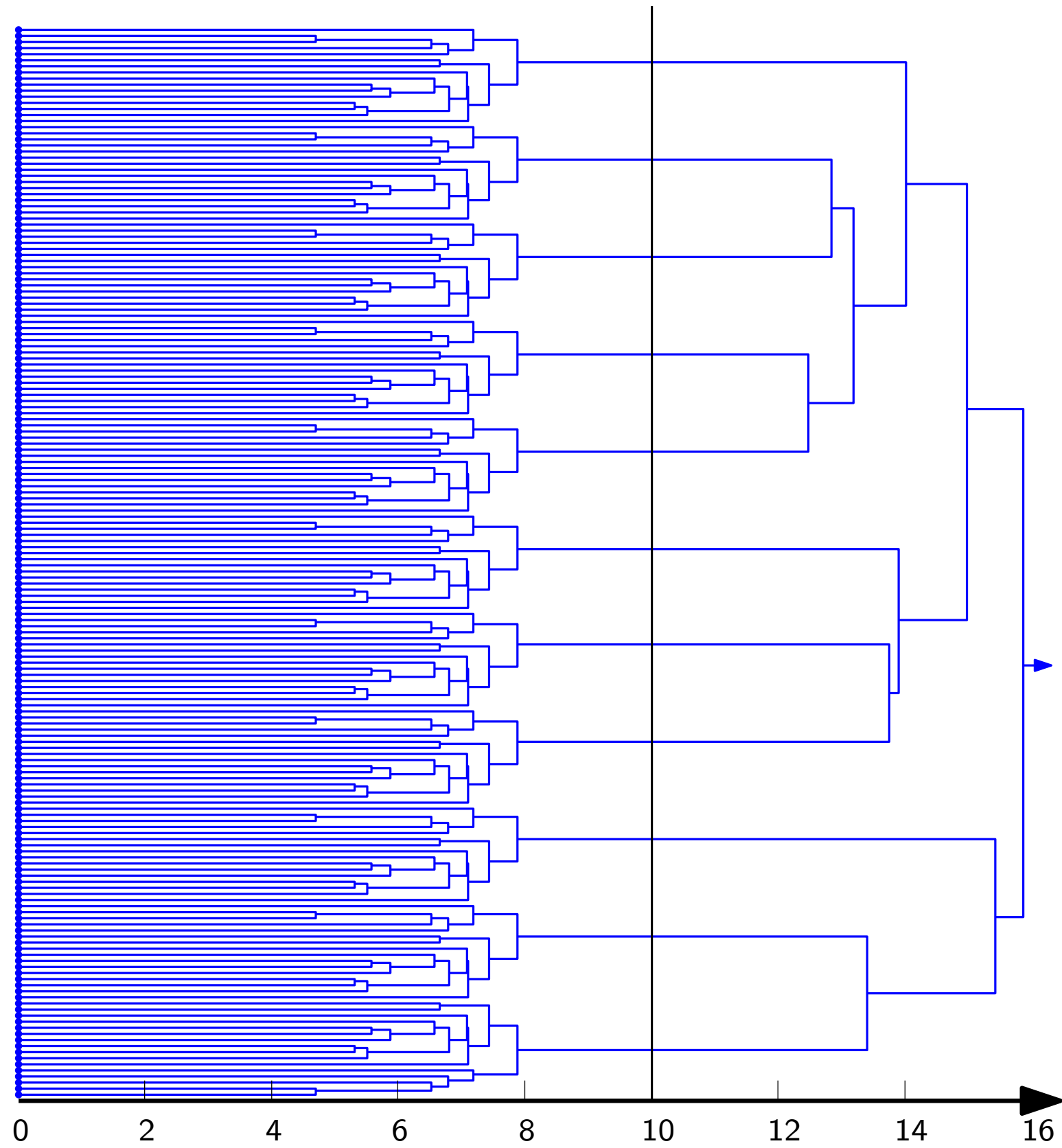
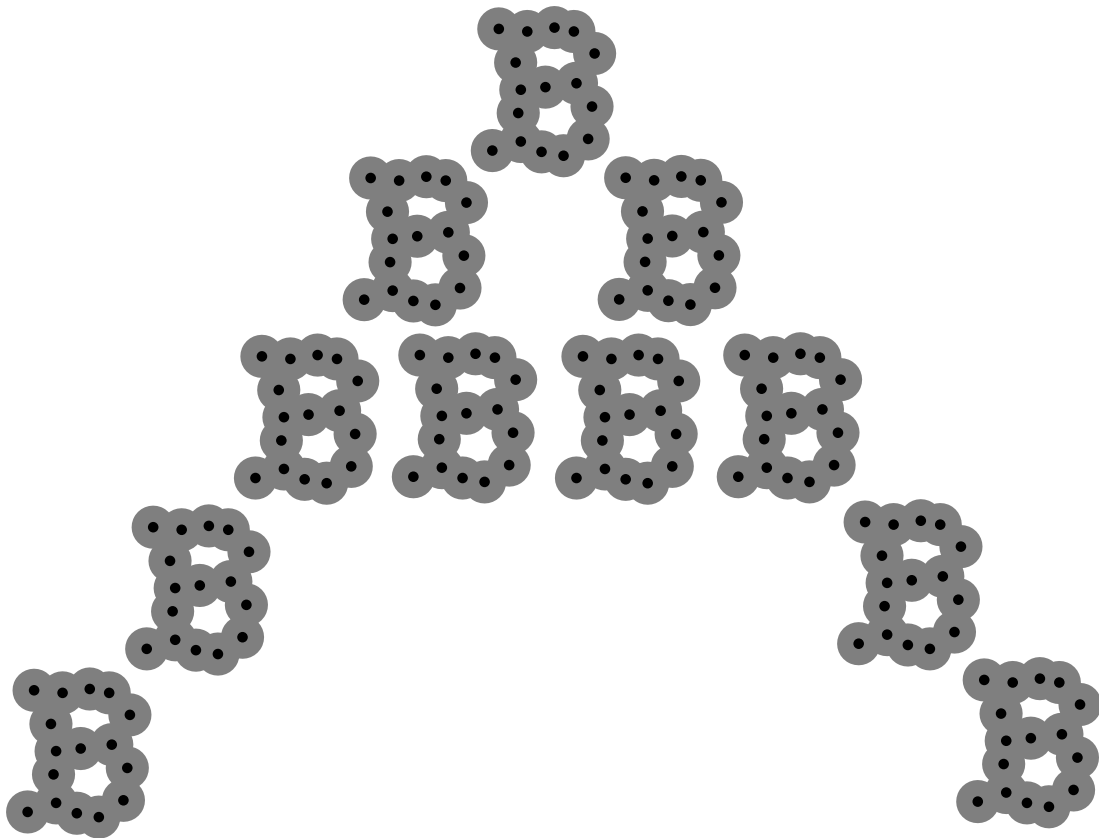
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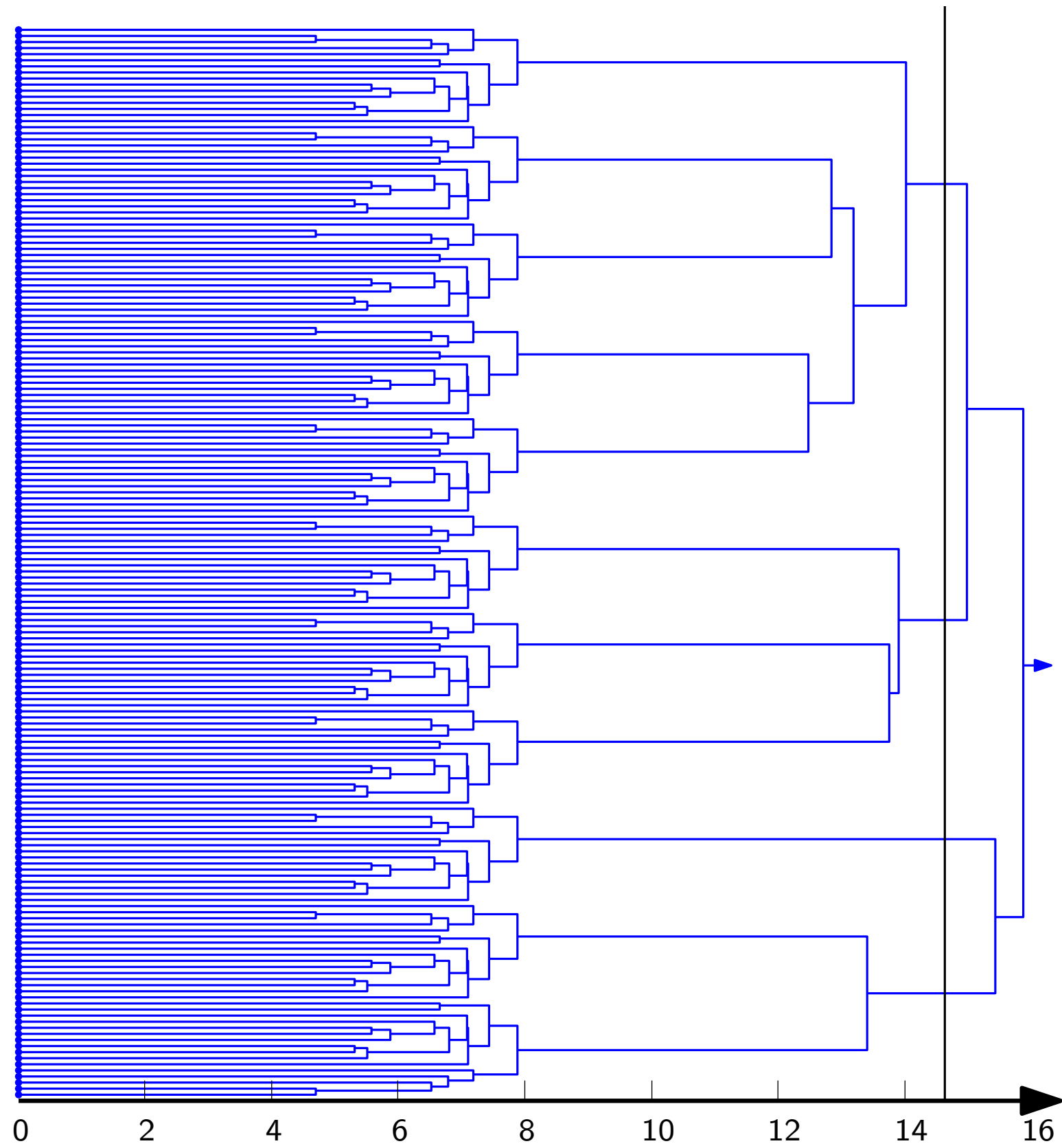
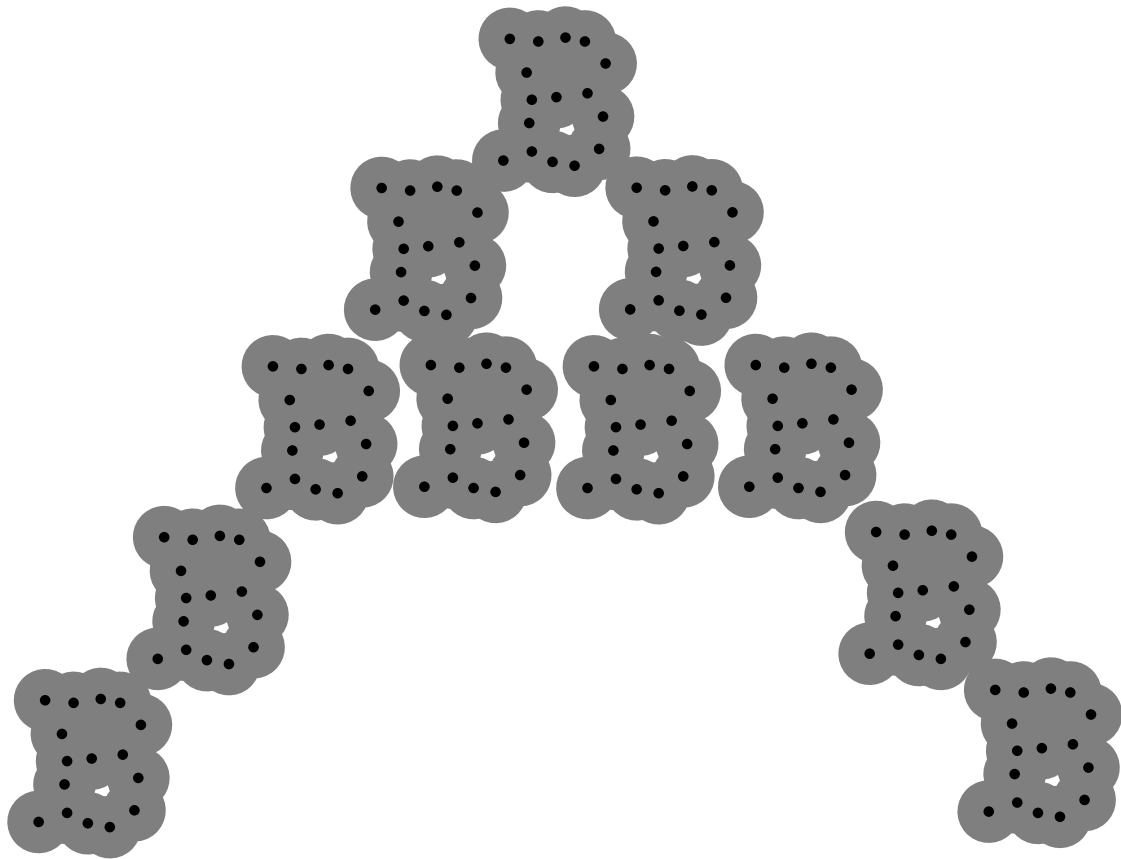
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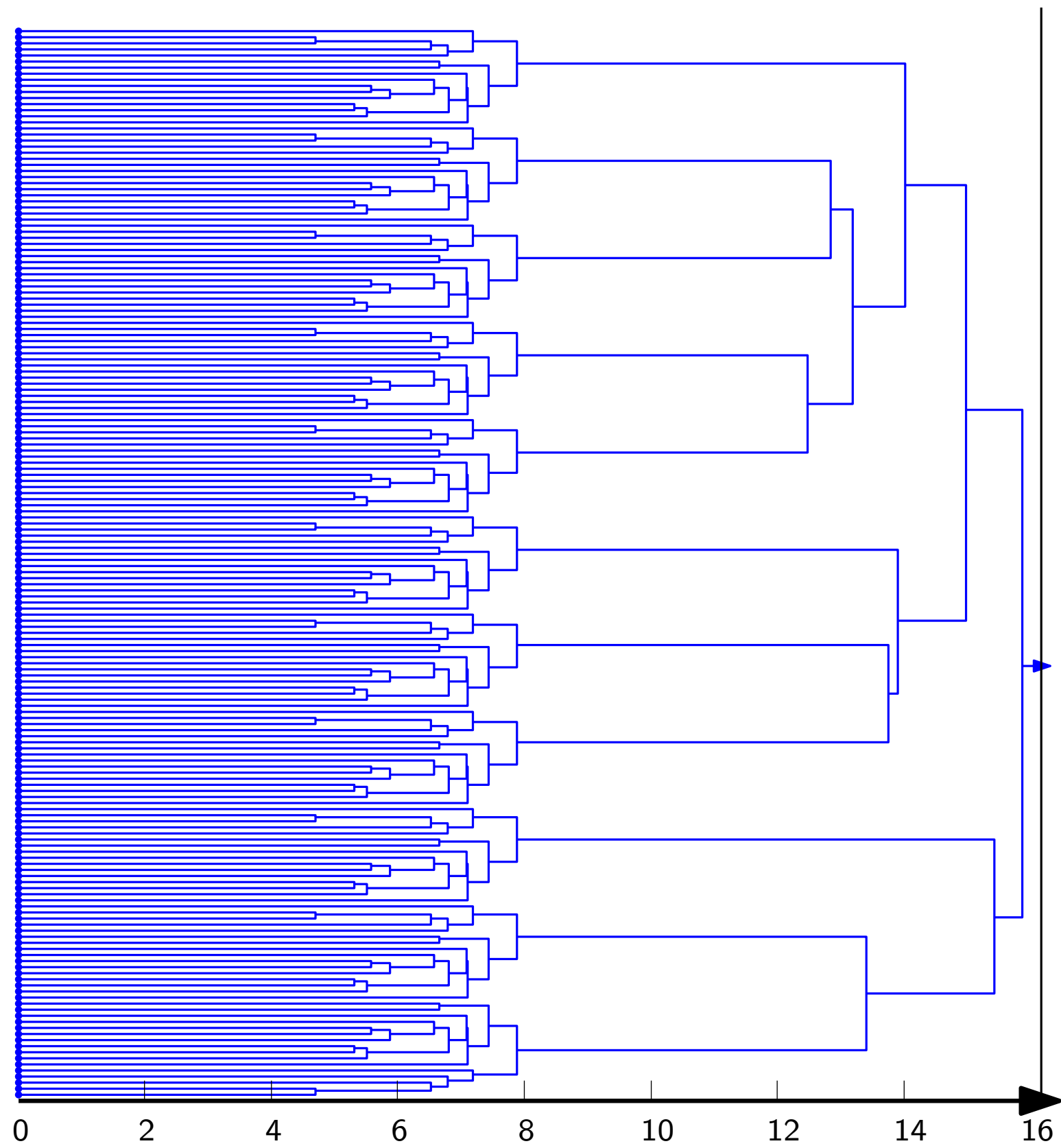
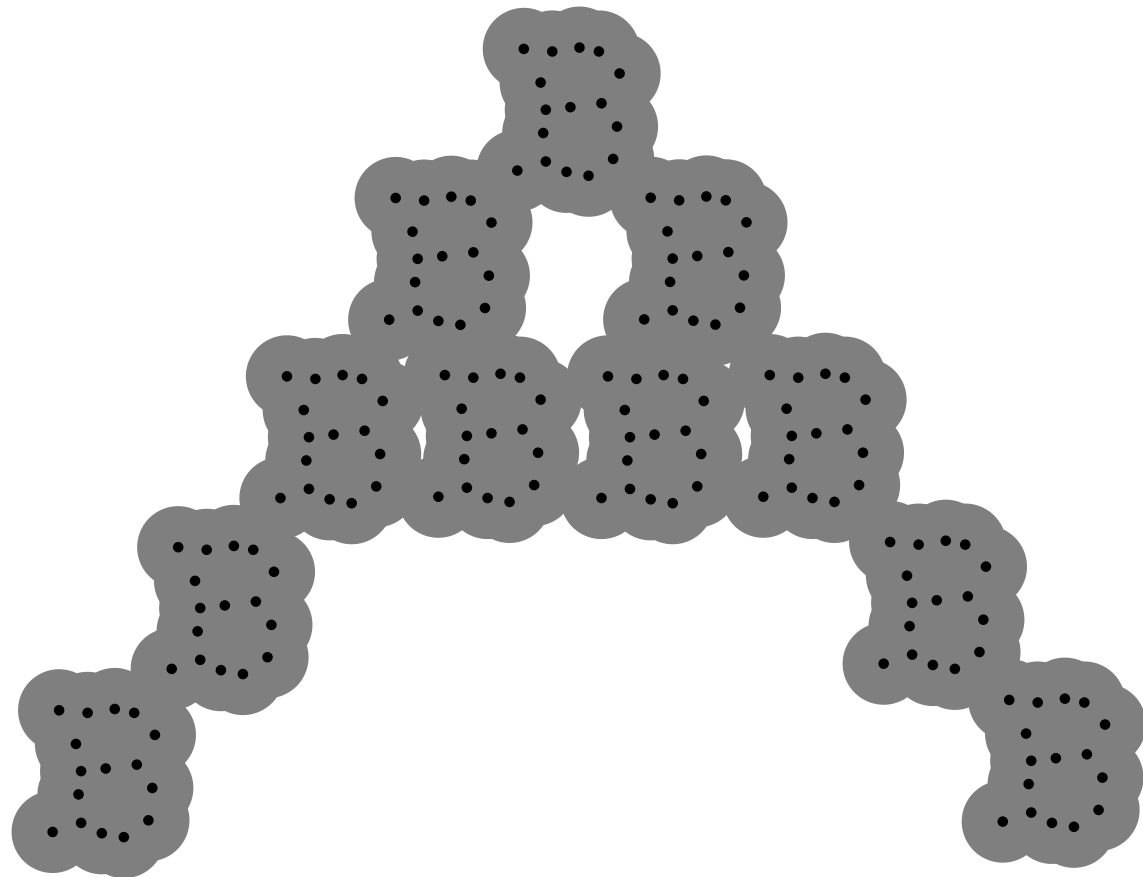
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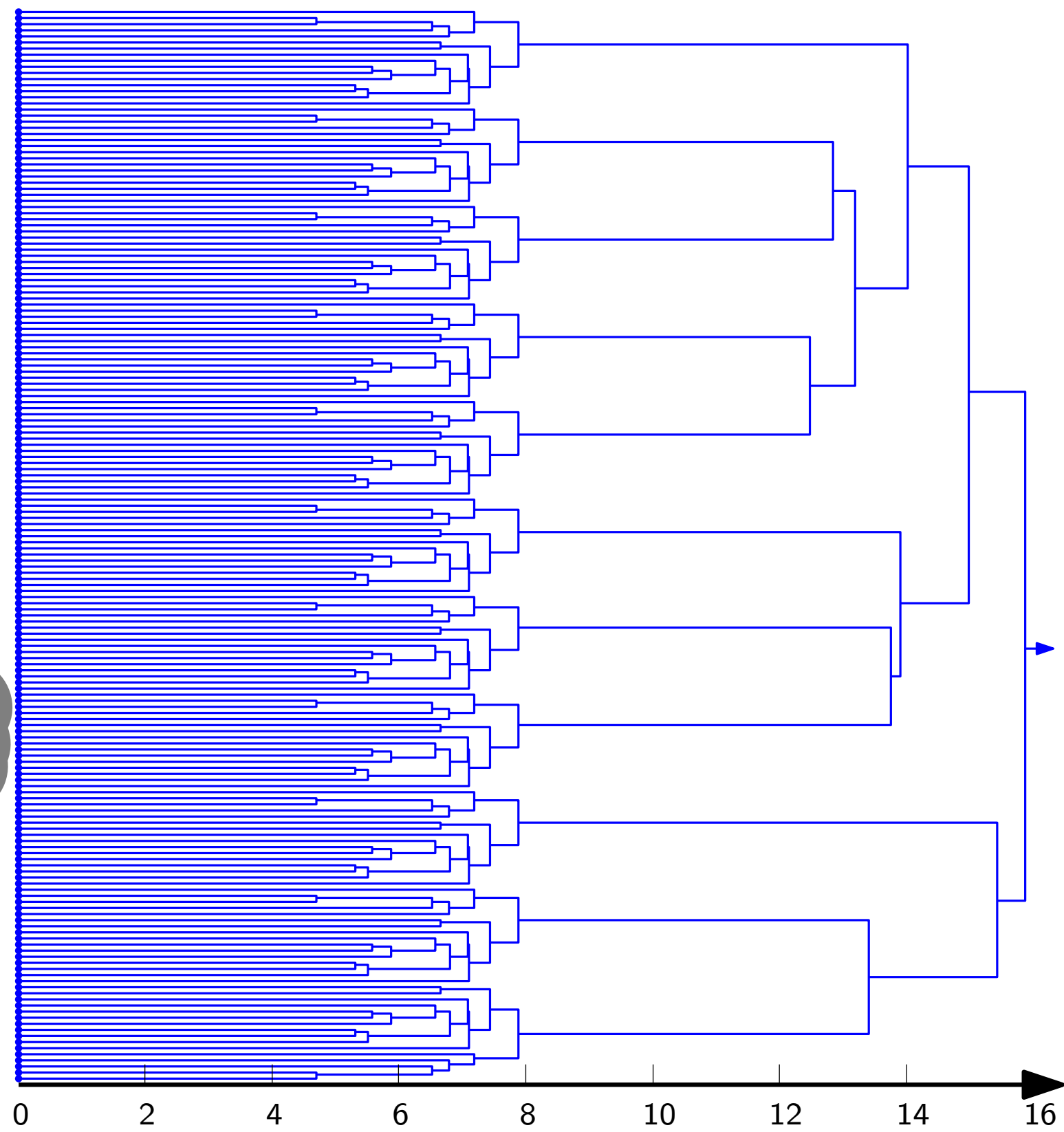
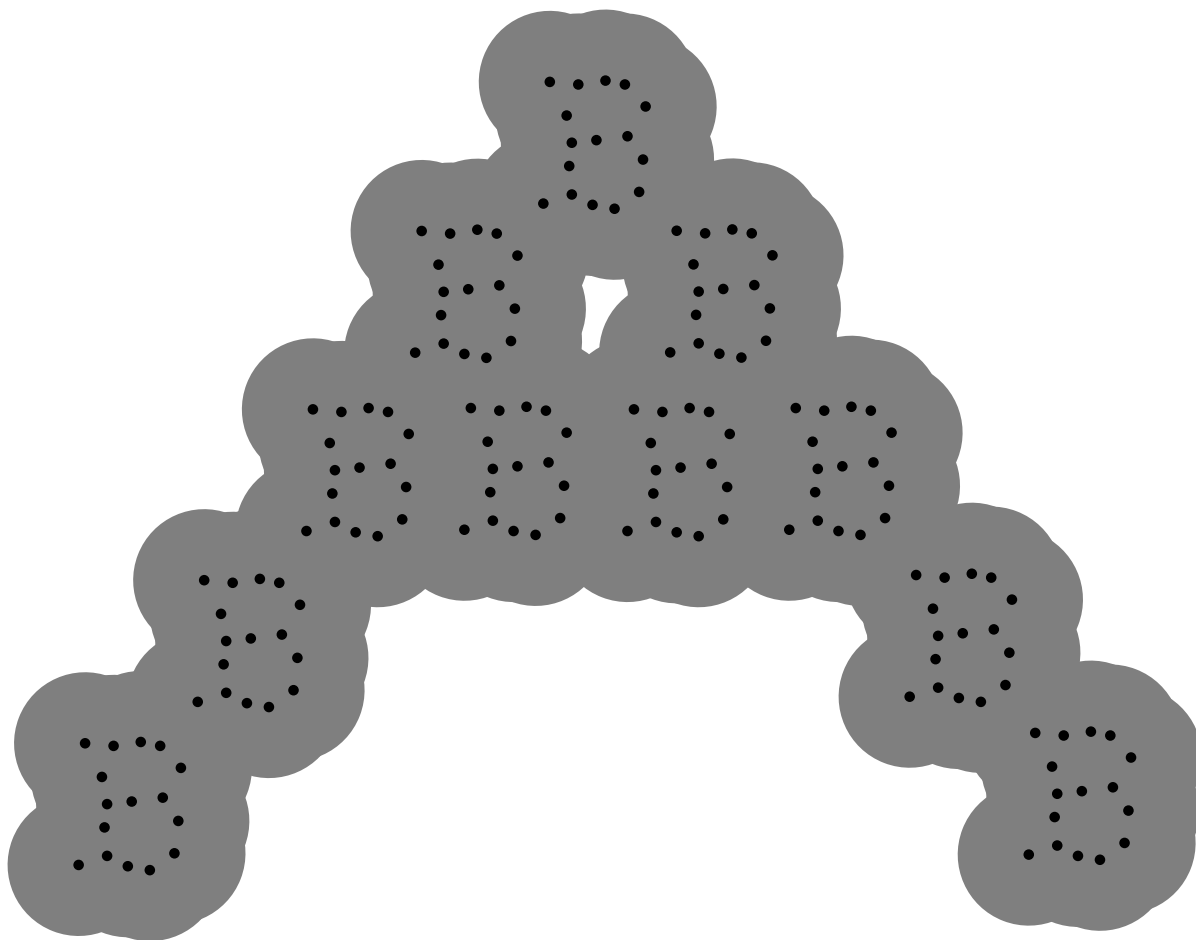
Distance Function

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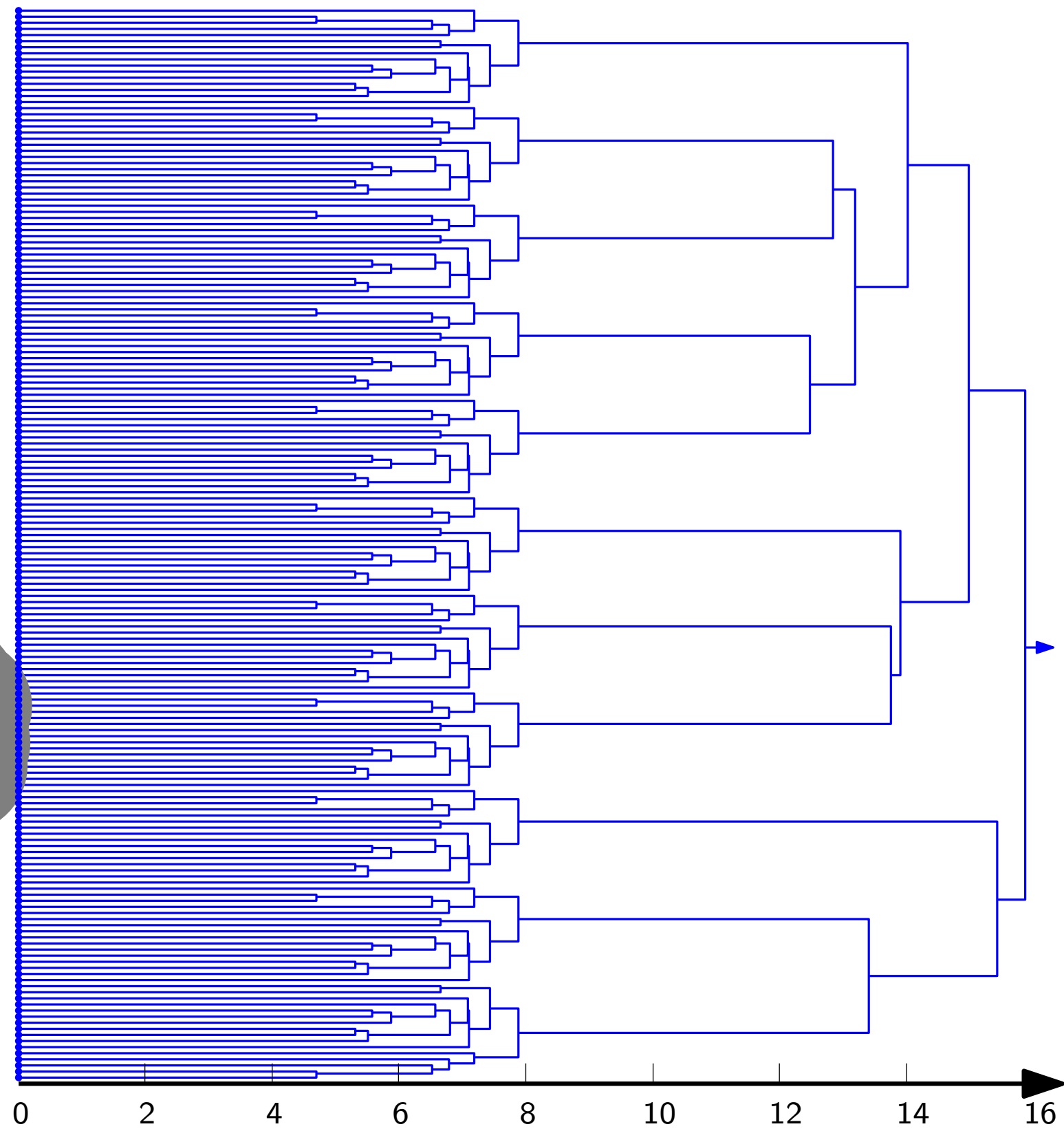
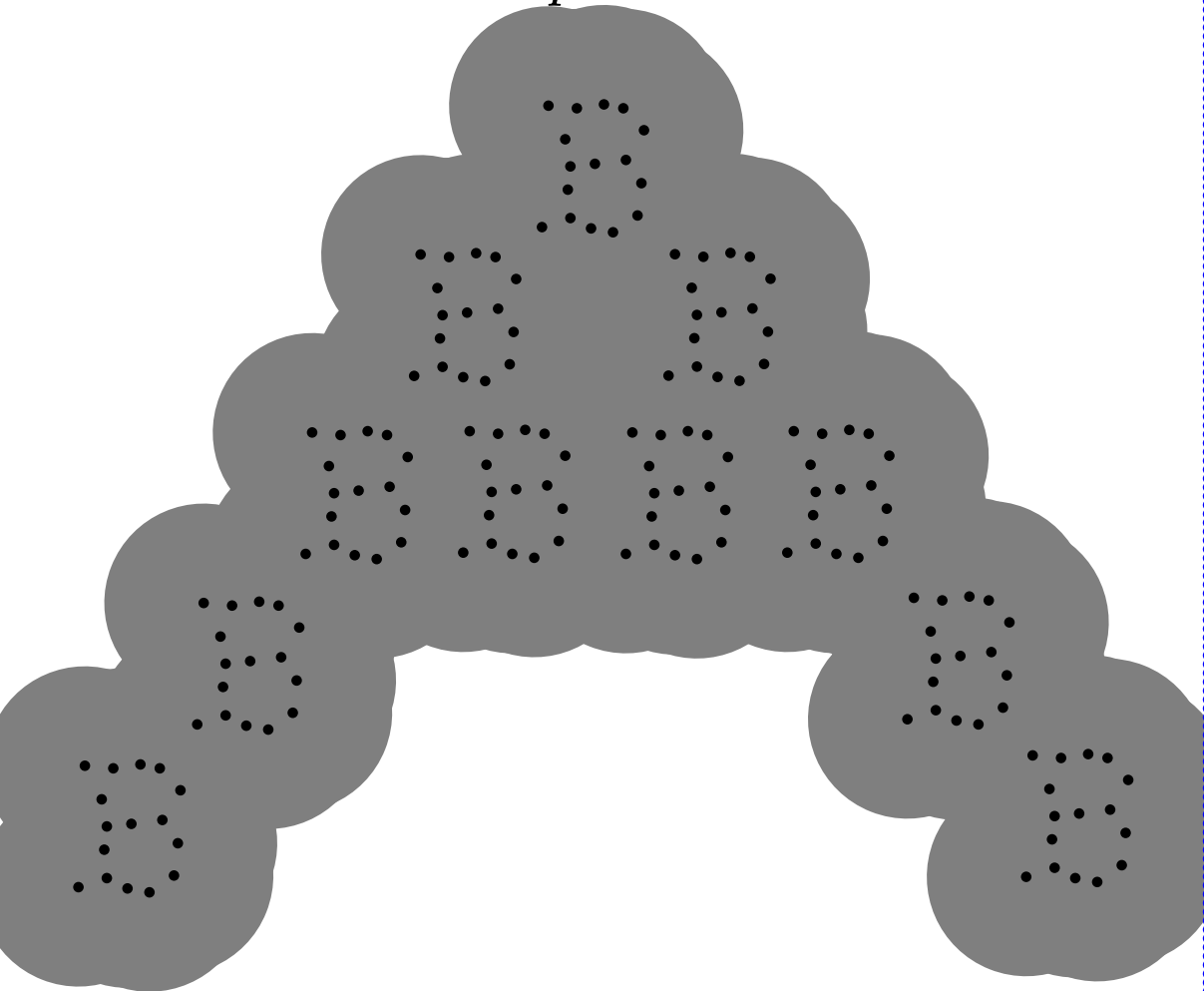
Distance Function

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Distance Function

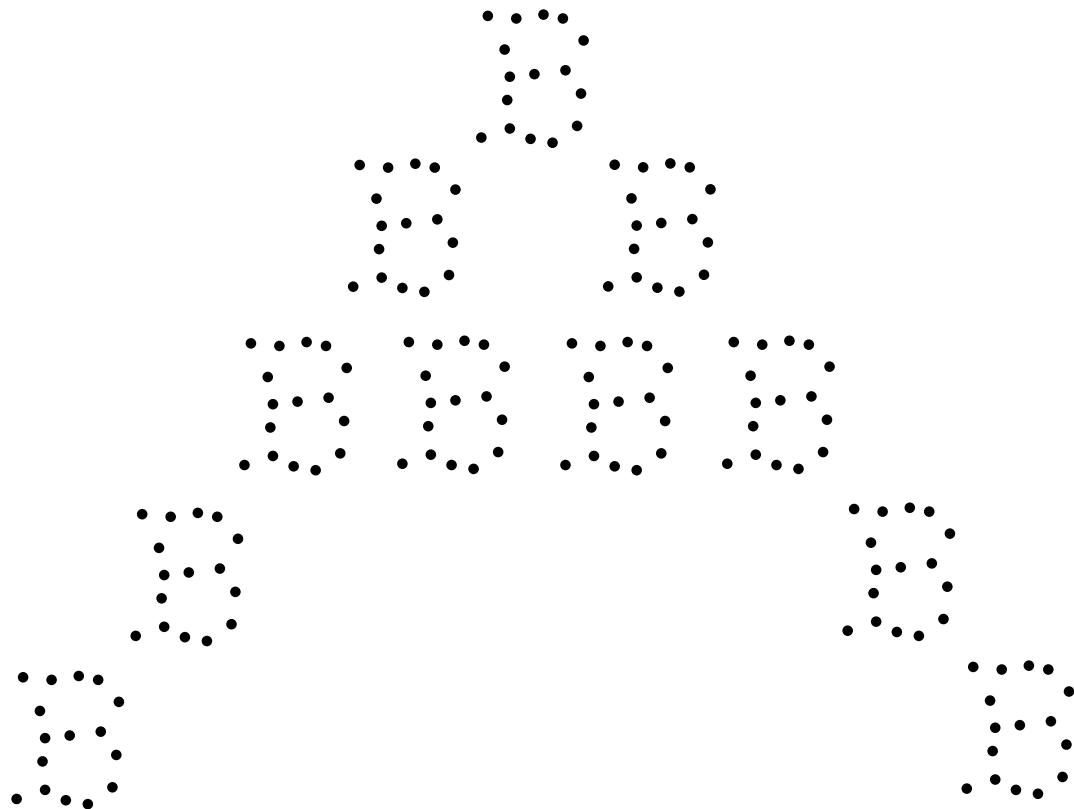
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Distance Function

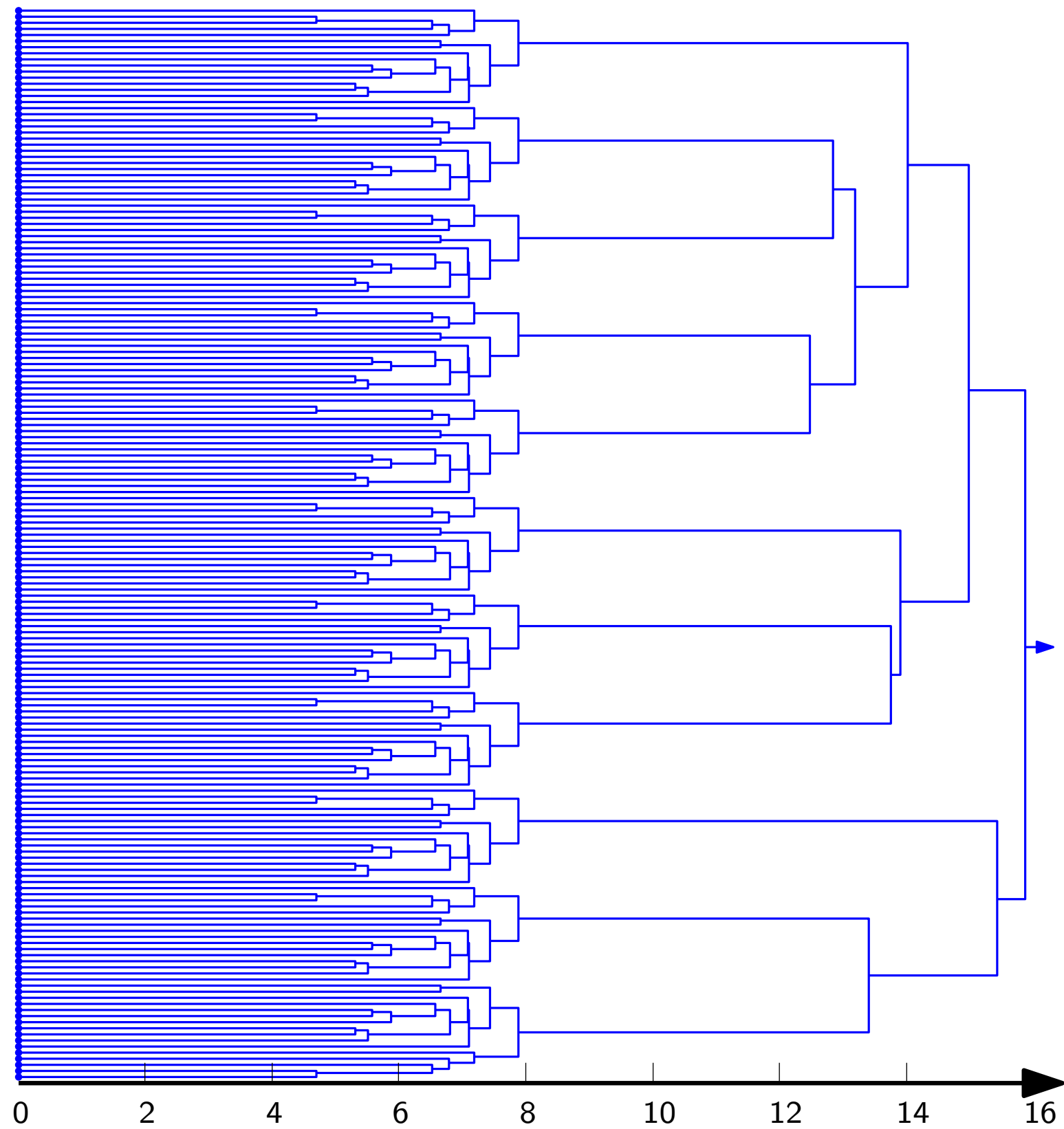
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dendrogram is:

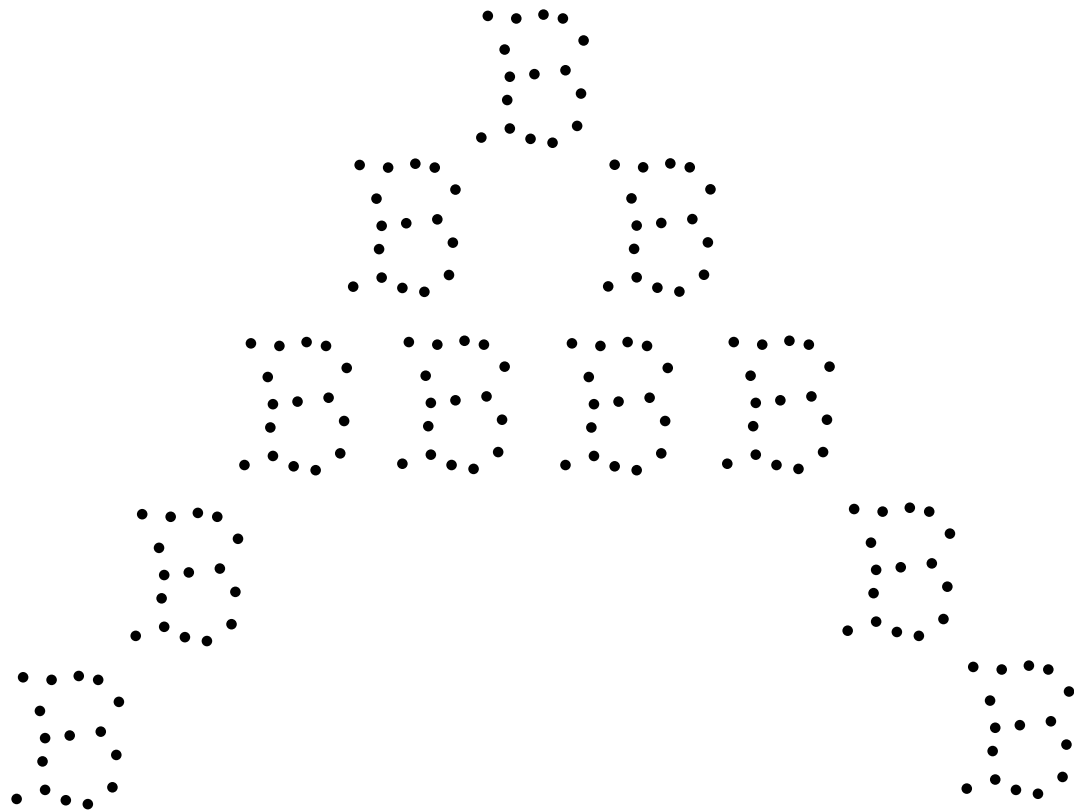
- informative
- unstable



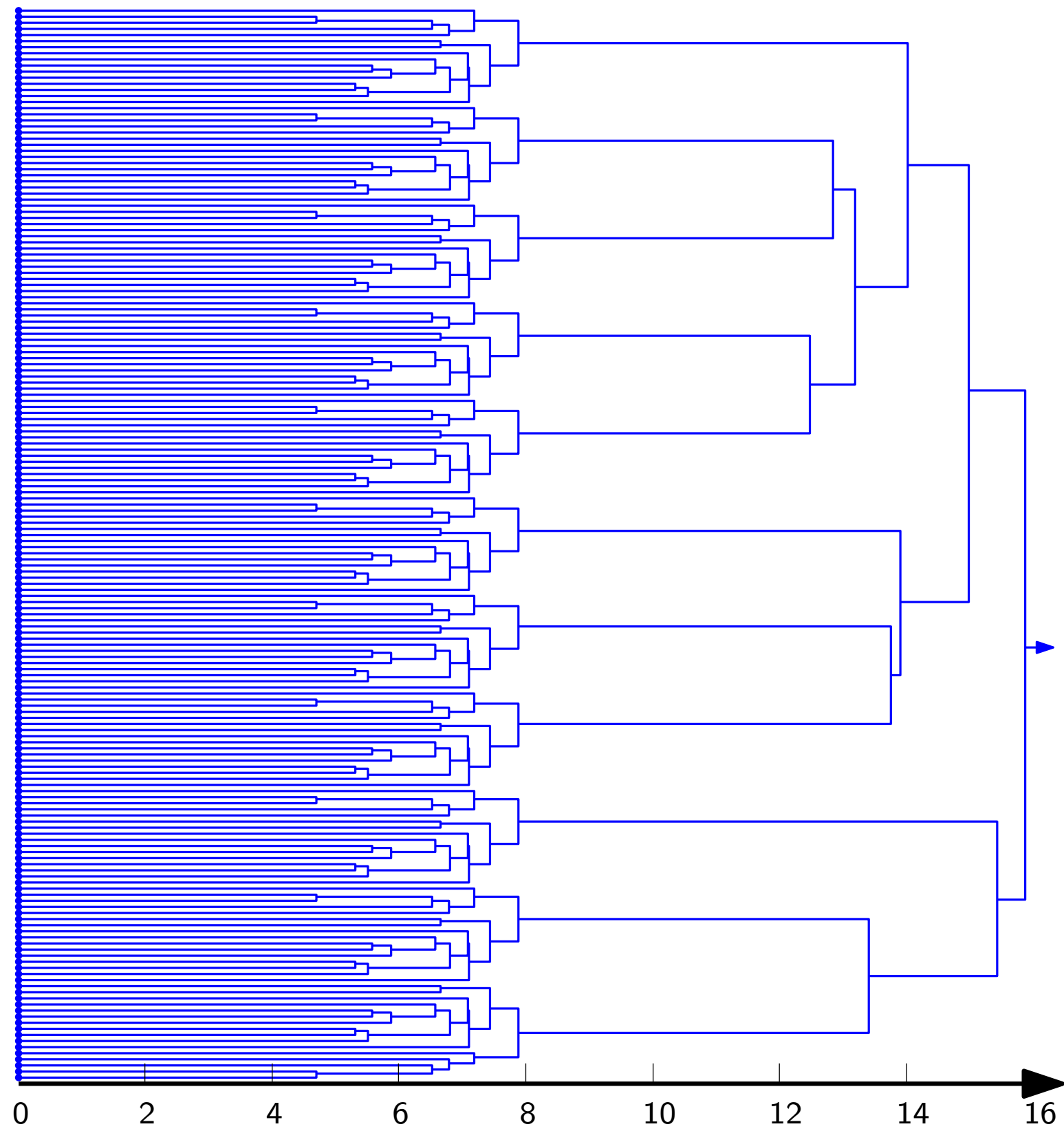
Distance Function

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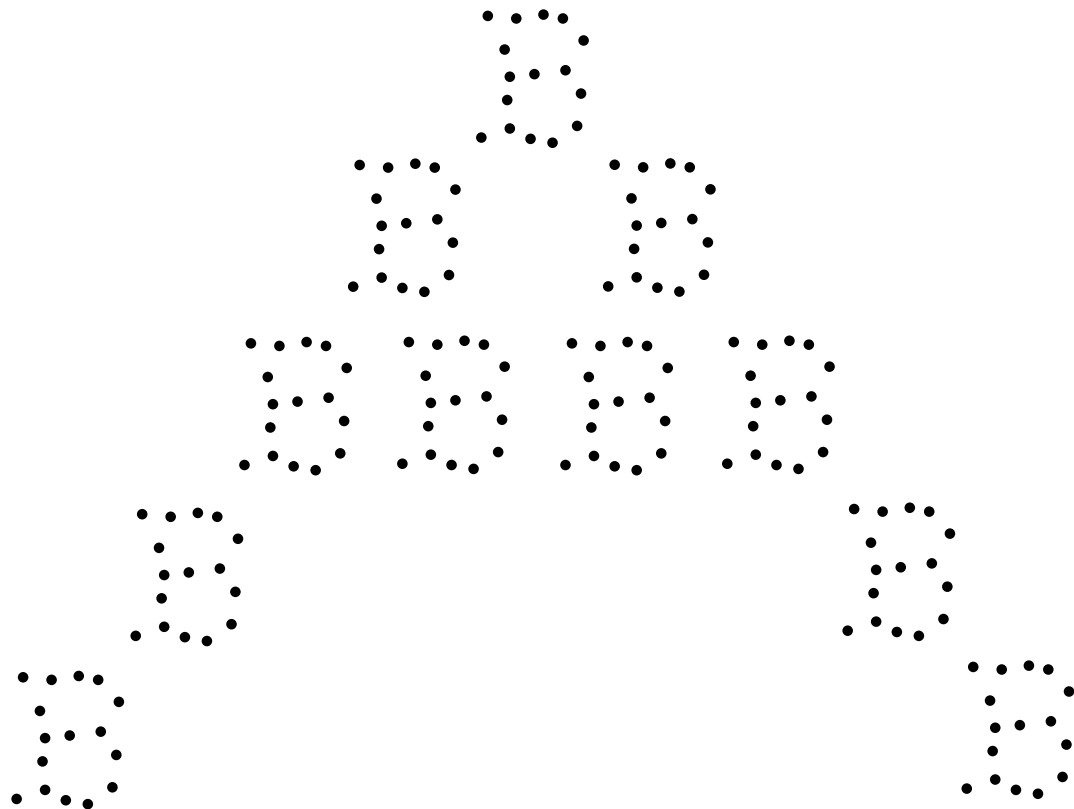
dendrogram $\rightarrow$ barcode



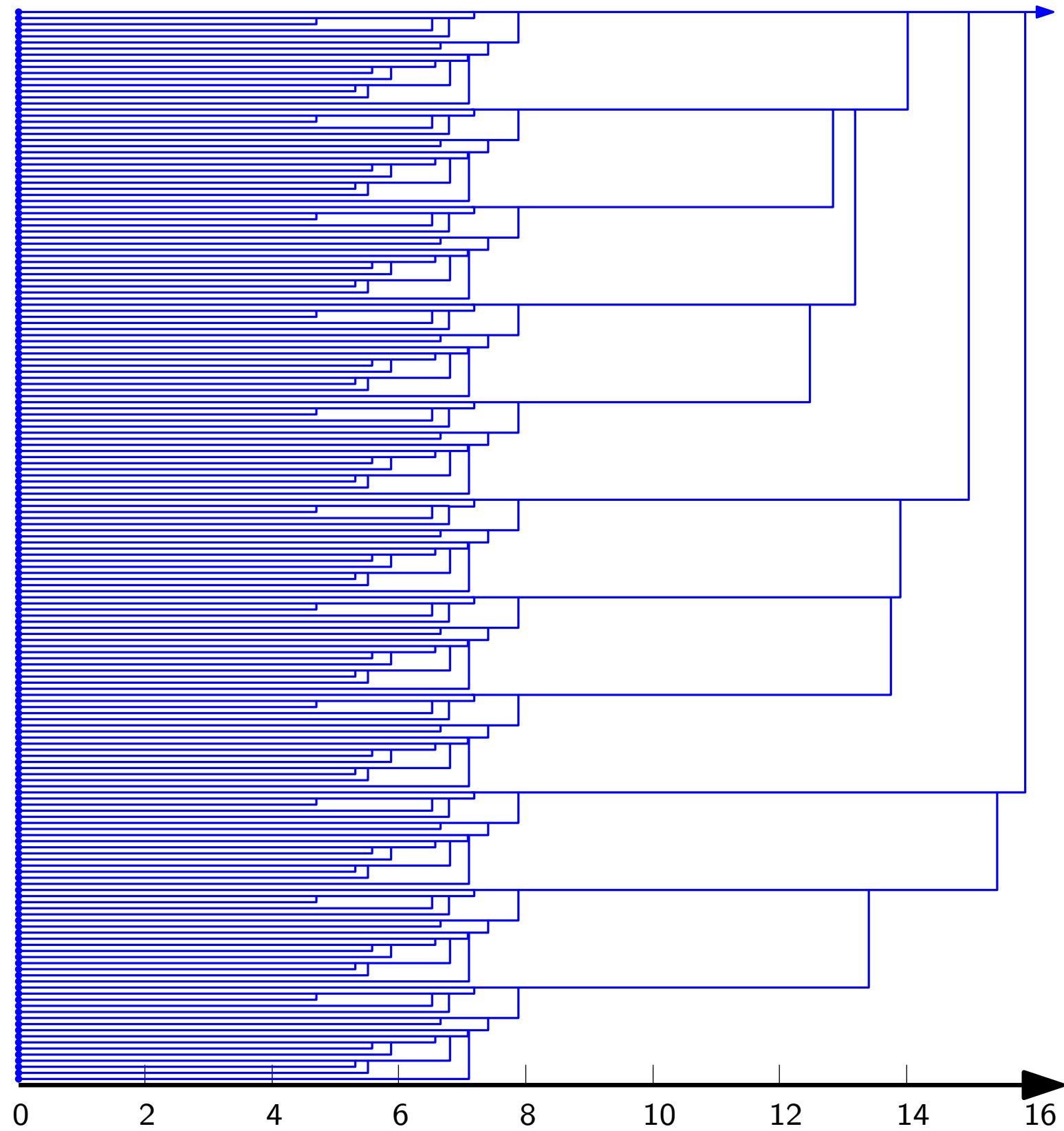
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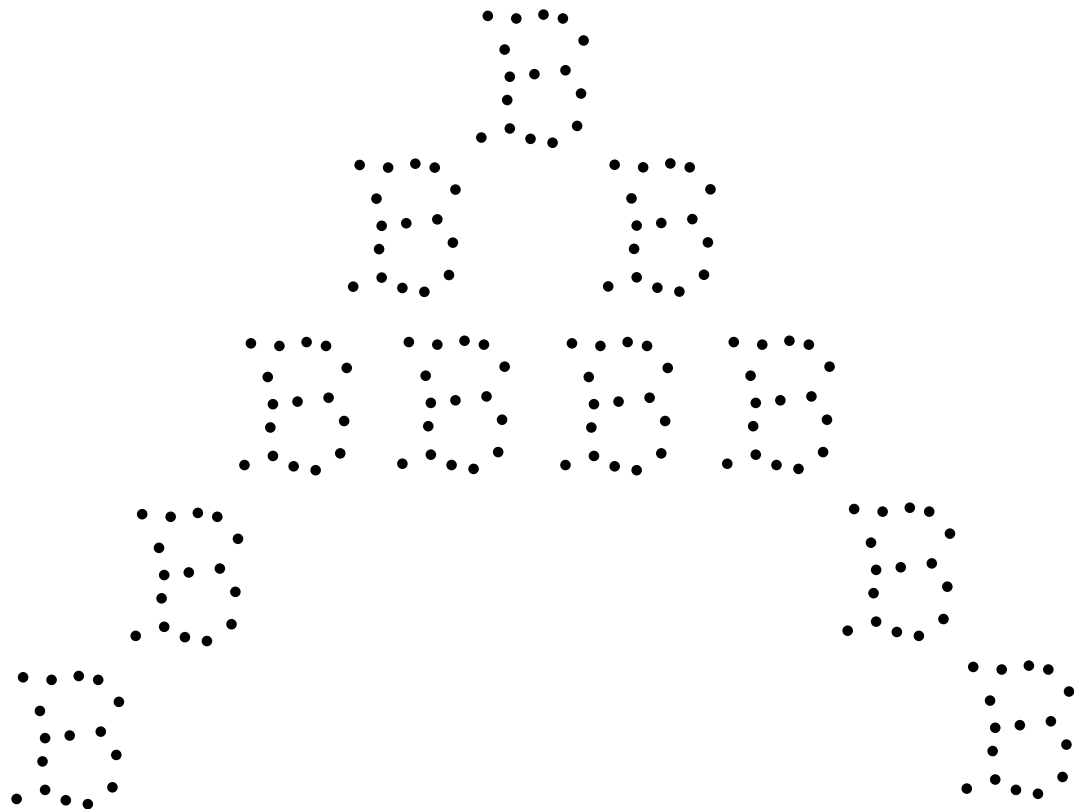
dendrogram $\rightarrow$ barcode



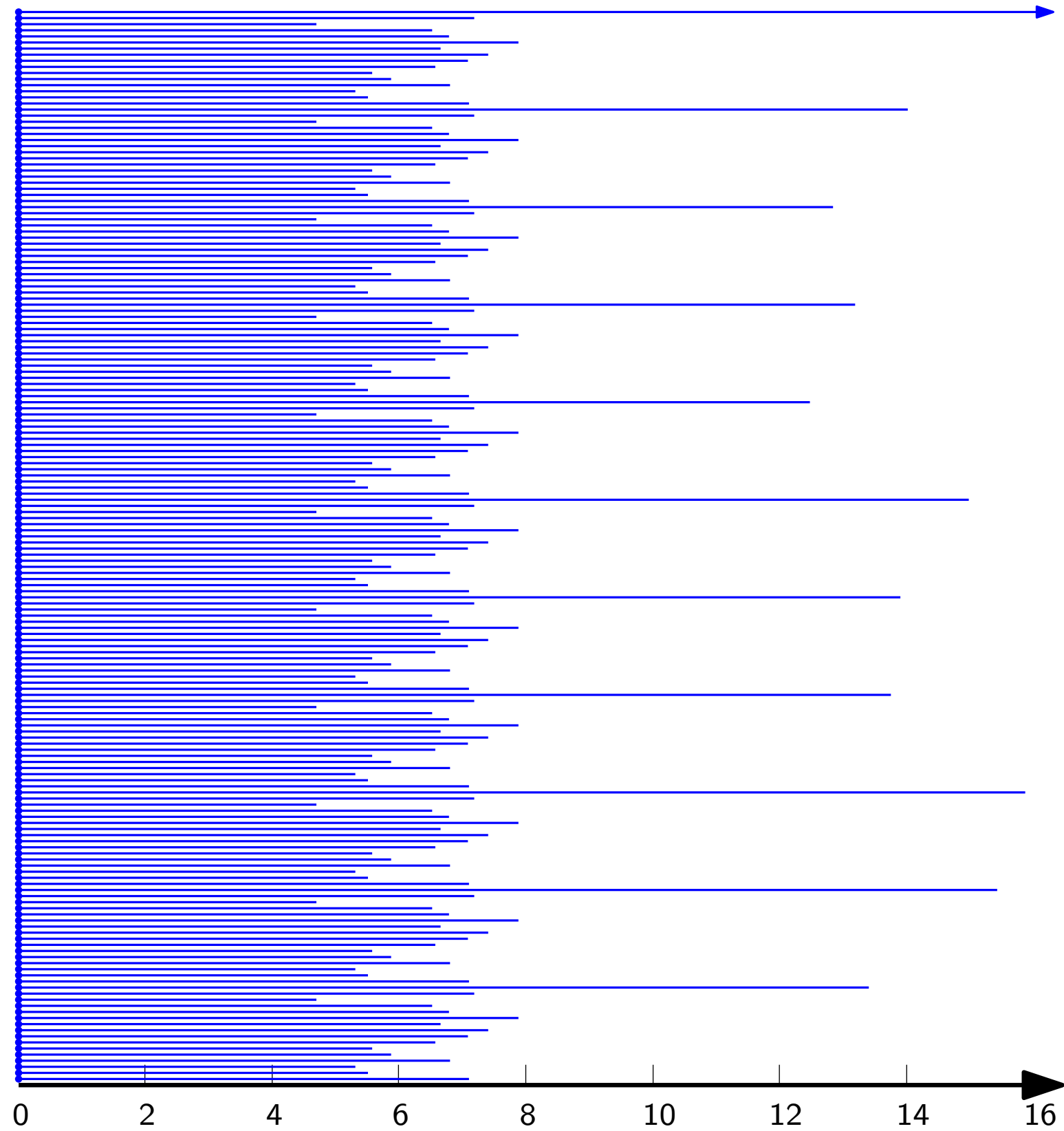
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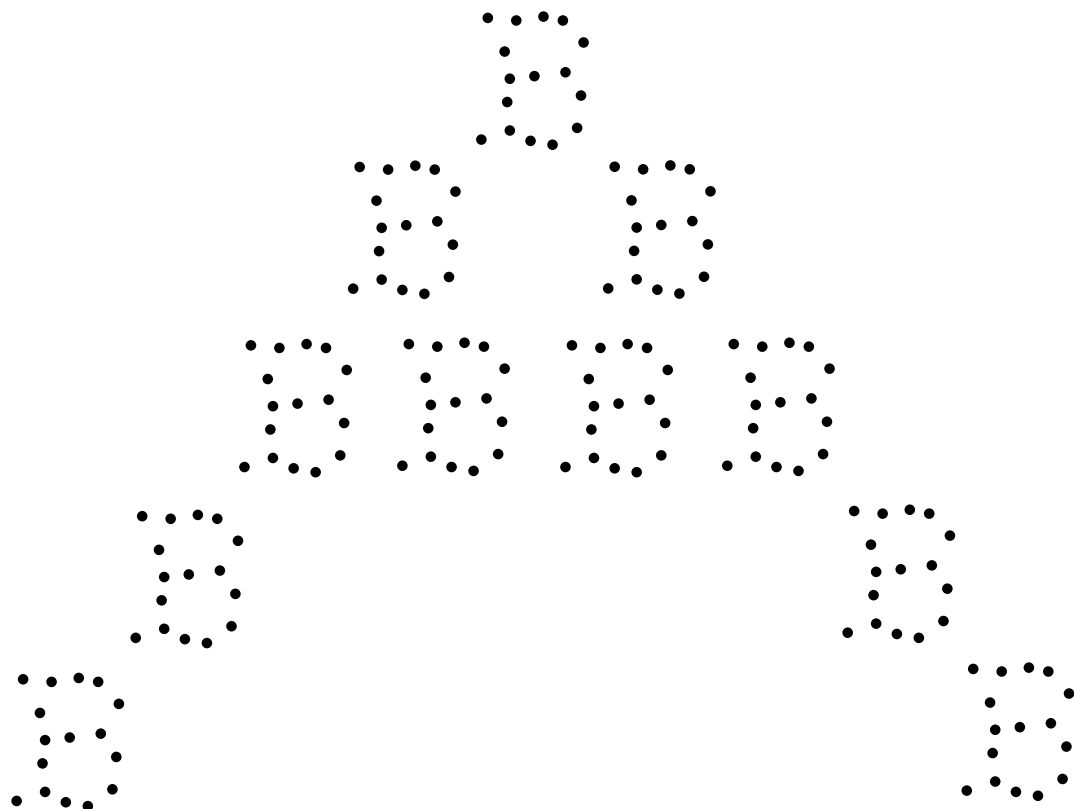
dendrogram $\rightarrow$ barcode



Distance Function

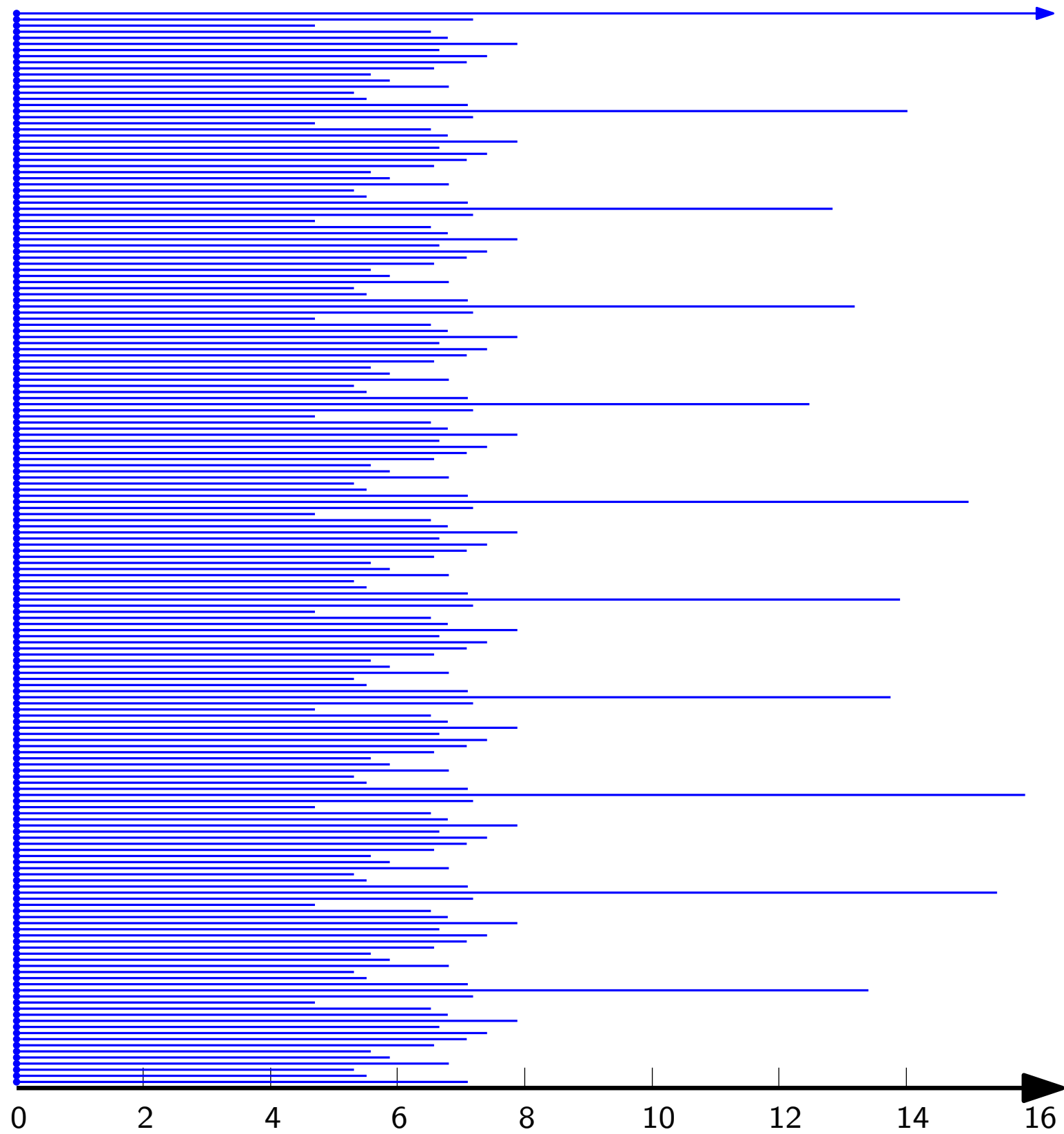
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barcode is:

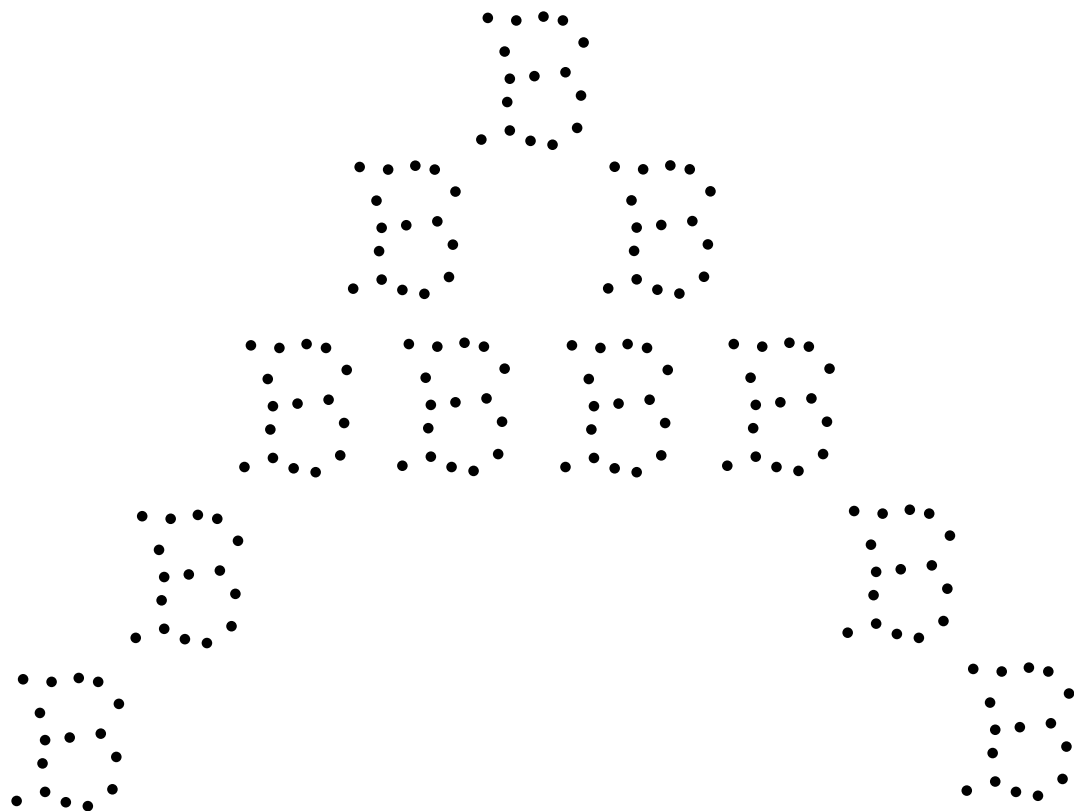
- less (but still) informative
- more stable



Distance Function

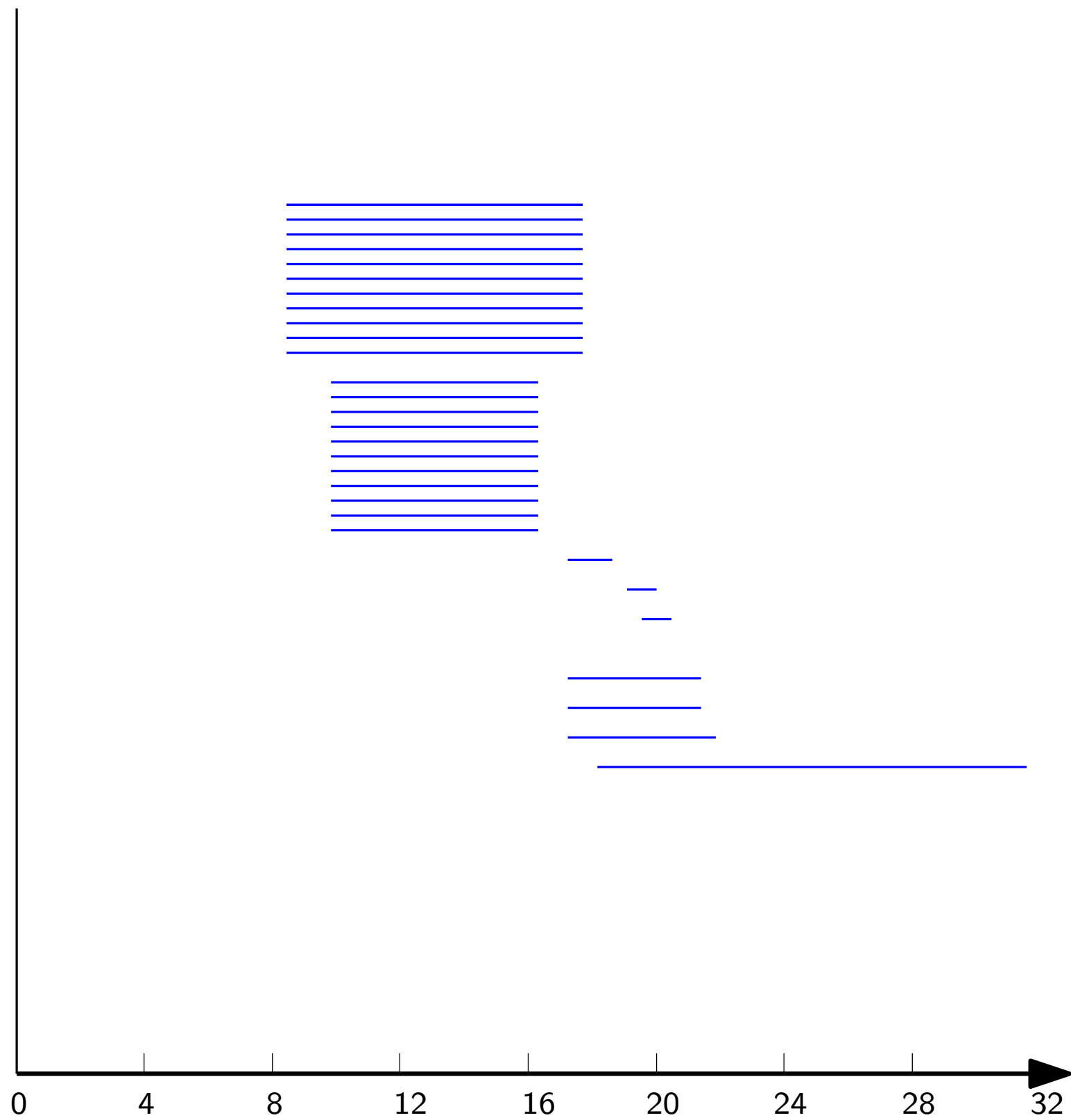
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barcode is:

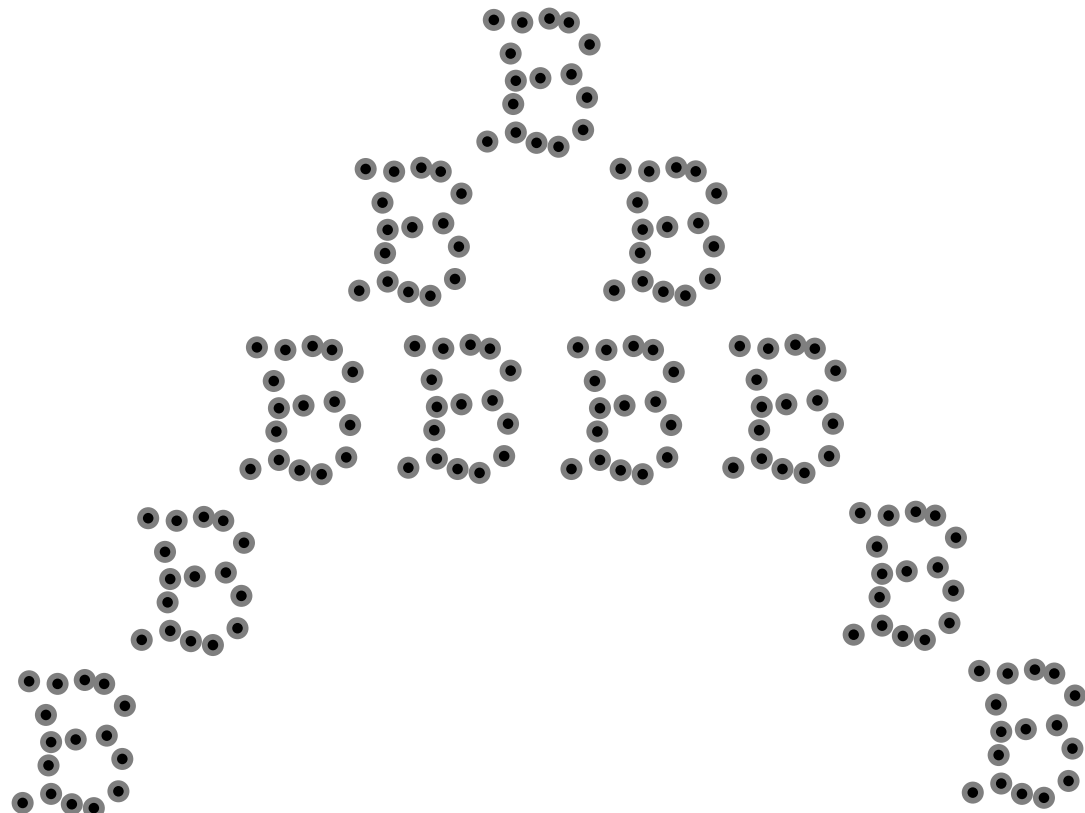
- less (but still) informative
- more stable
- generalizable



Distance Function

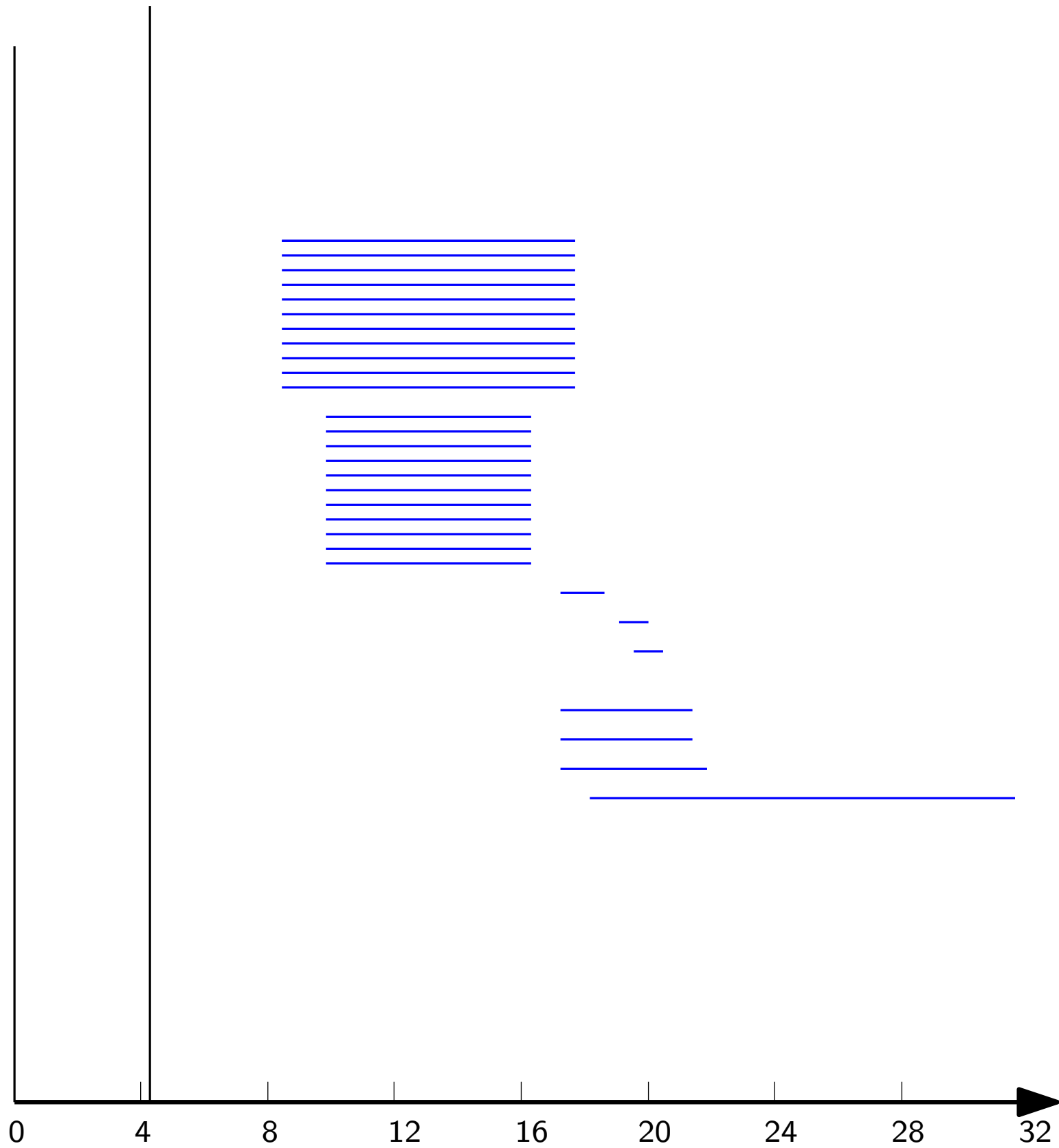
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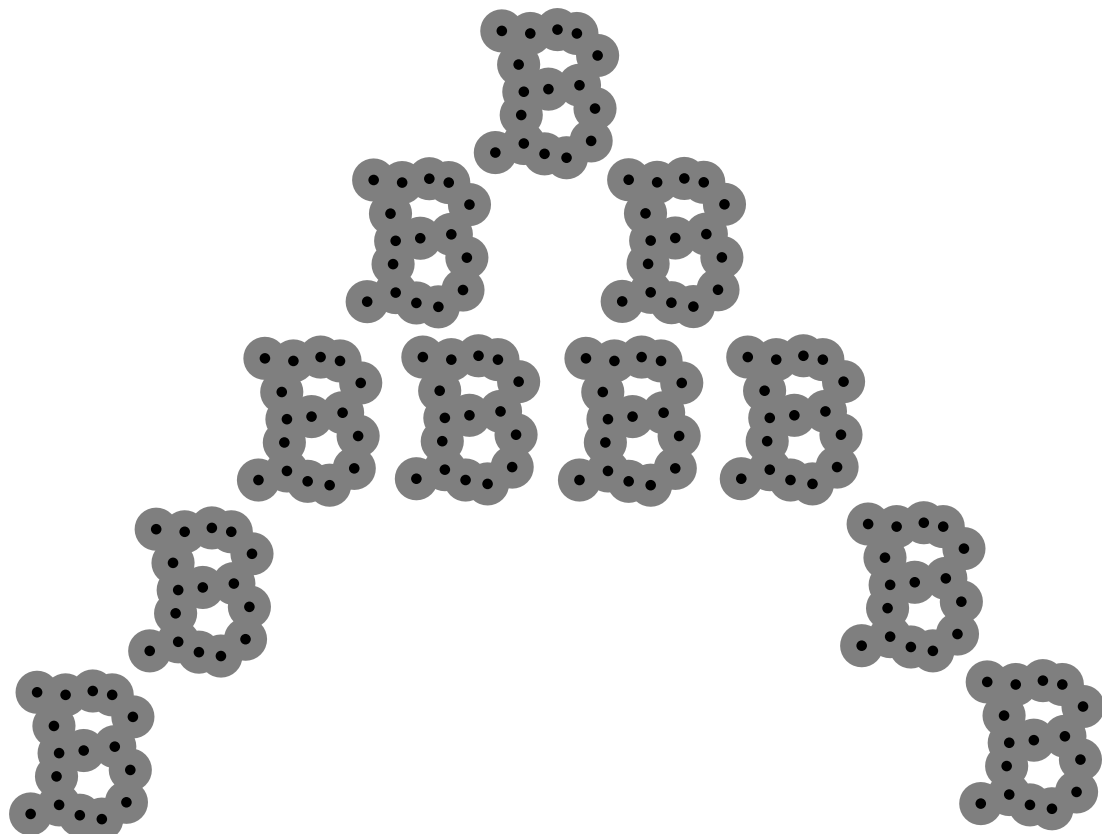
- less (but still) informative
- more stable
- generalizable



Distance Function

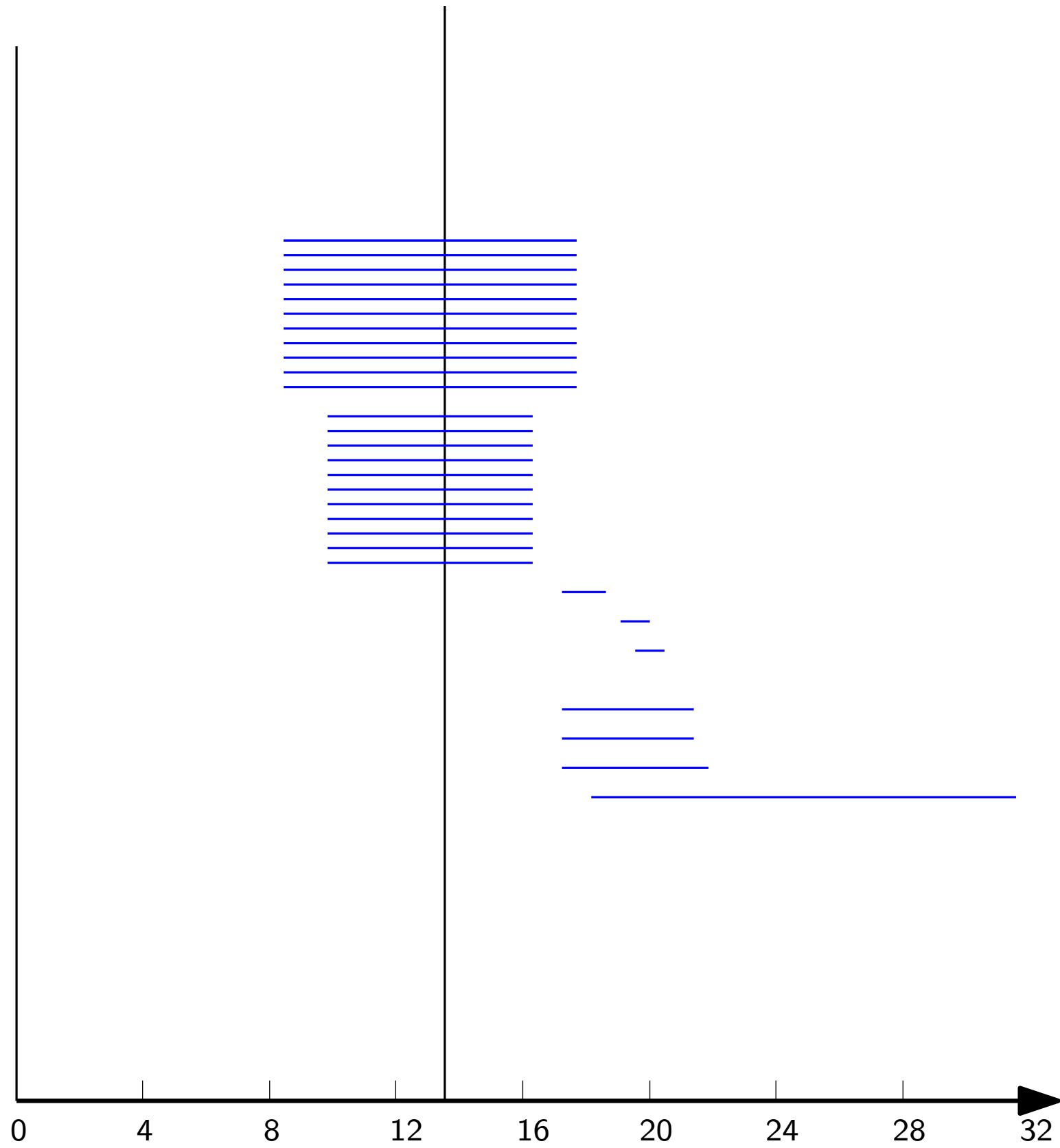
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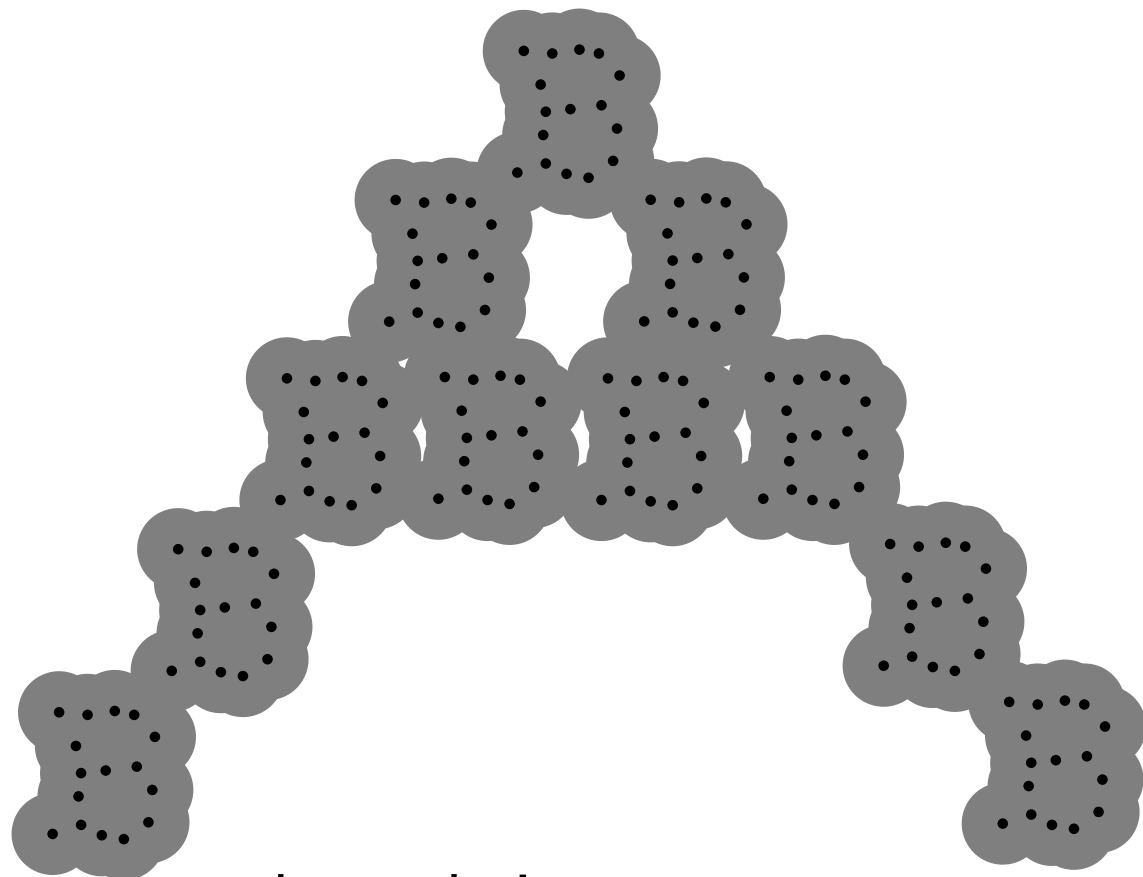
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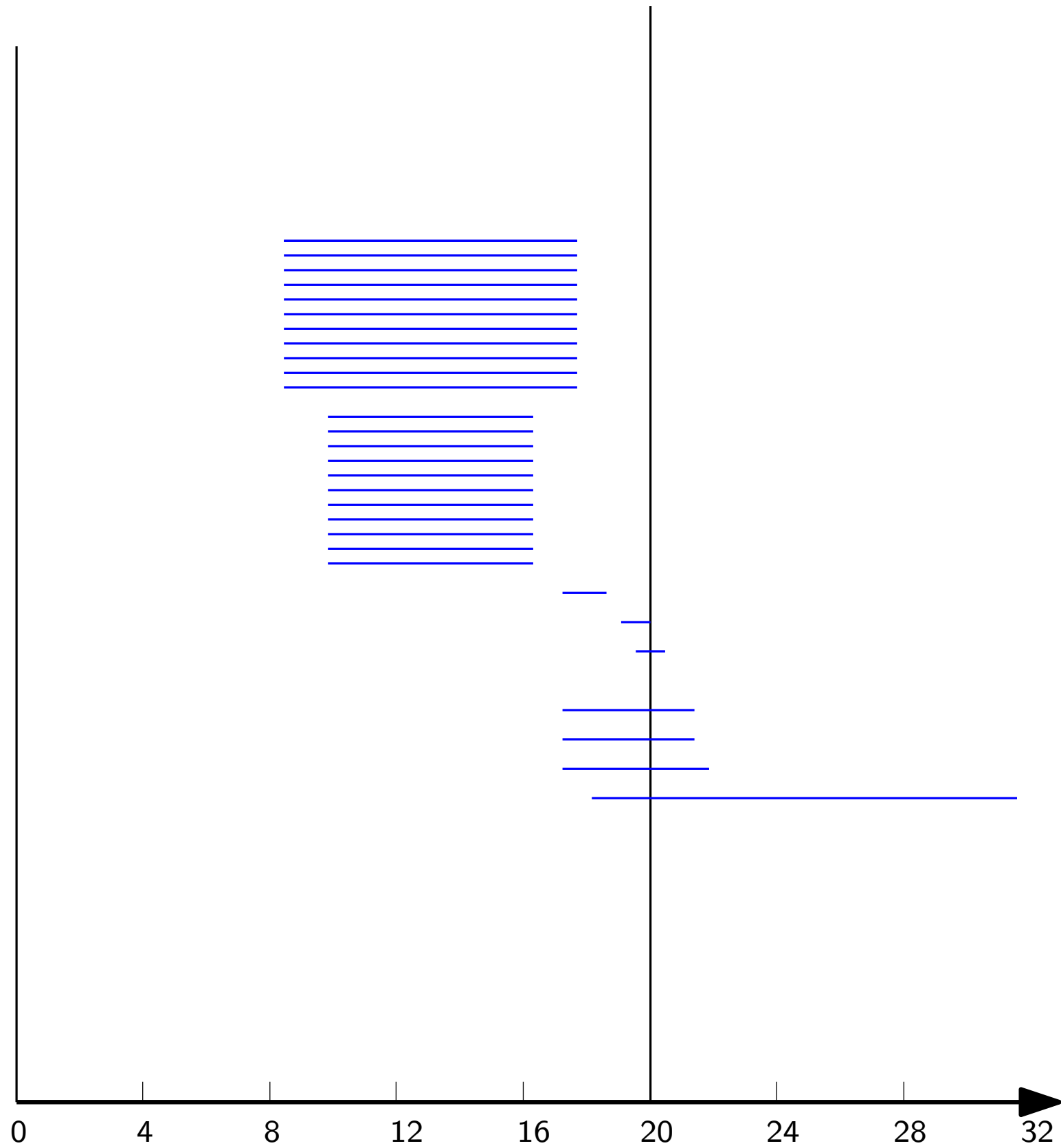
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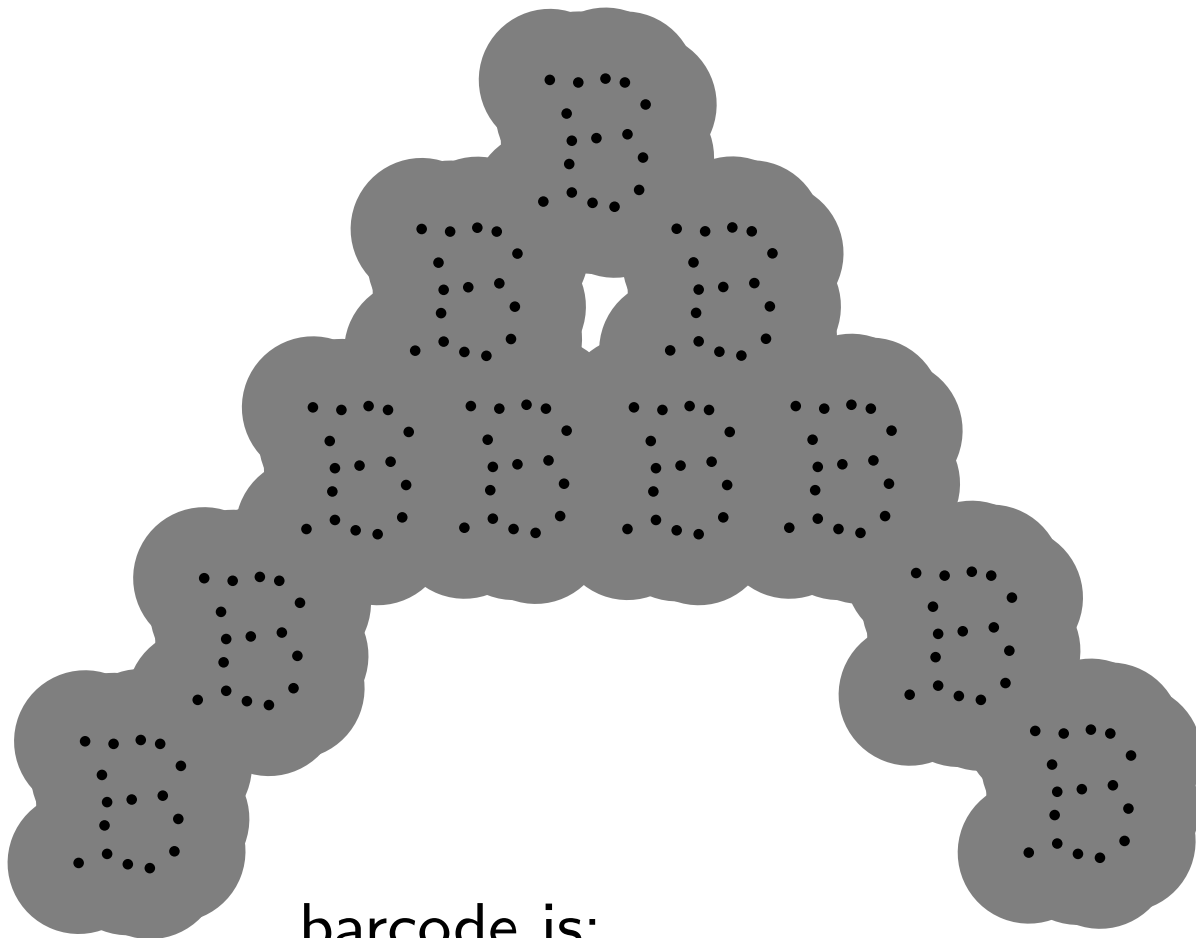
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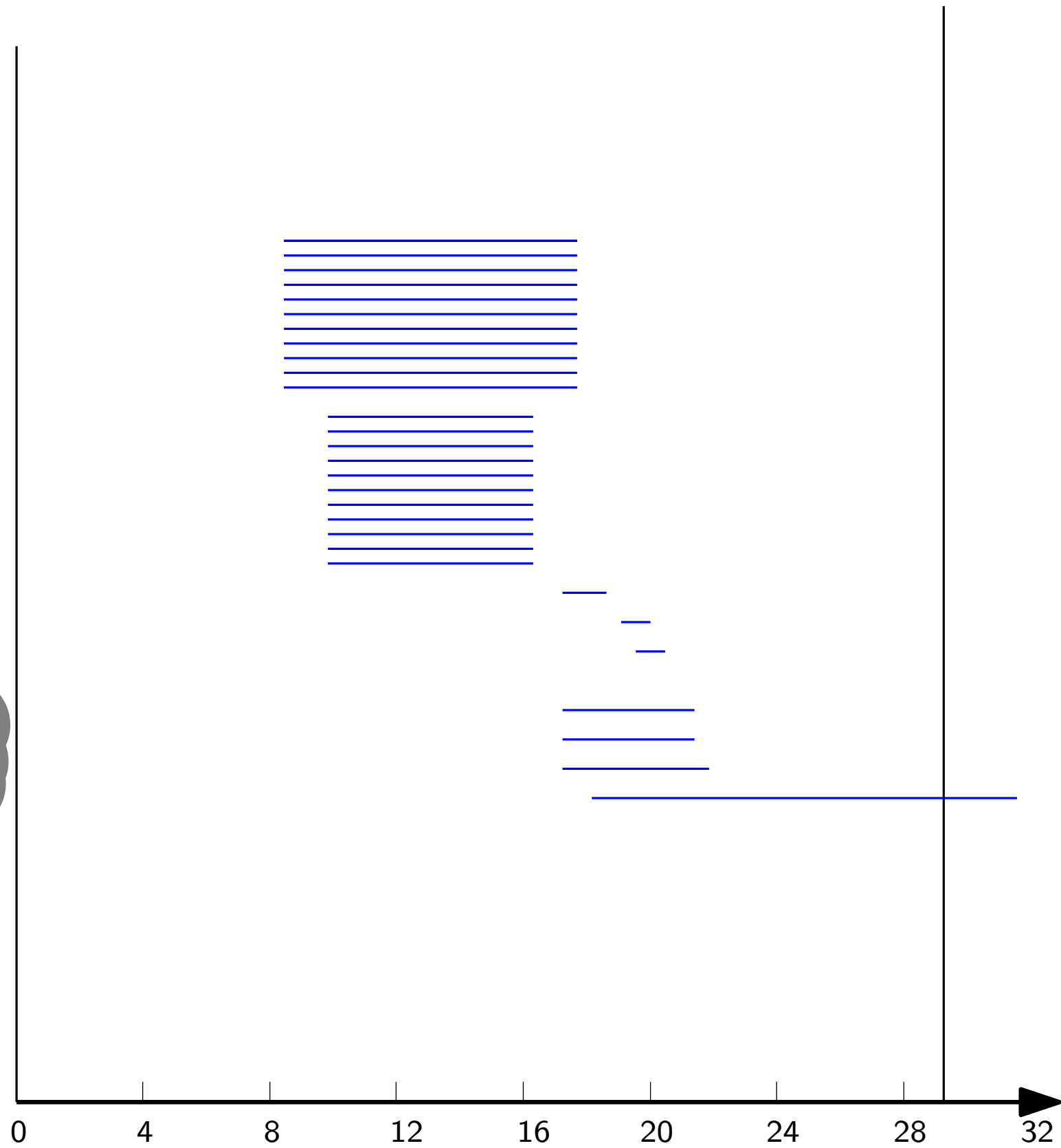
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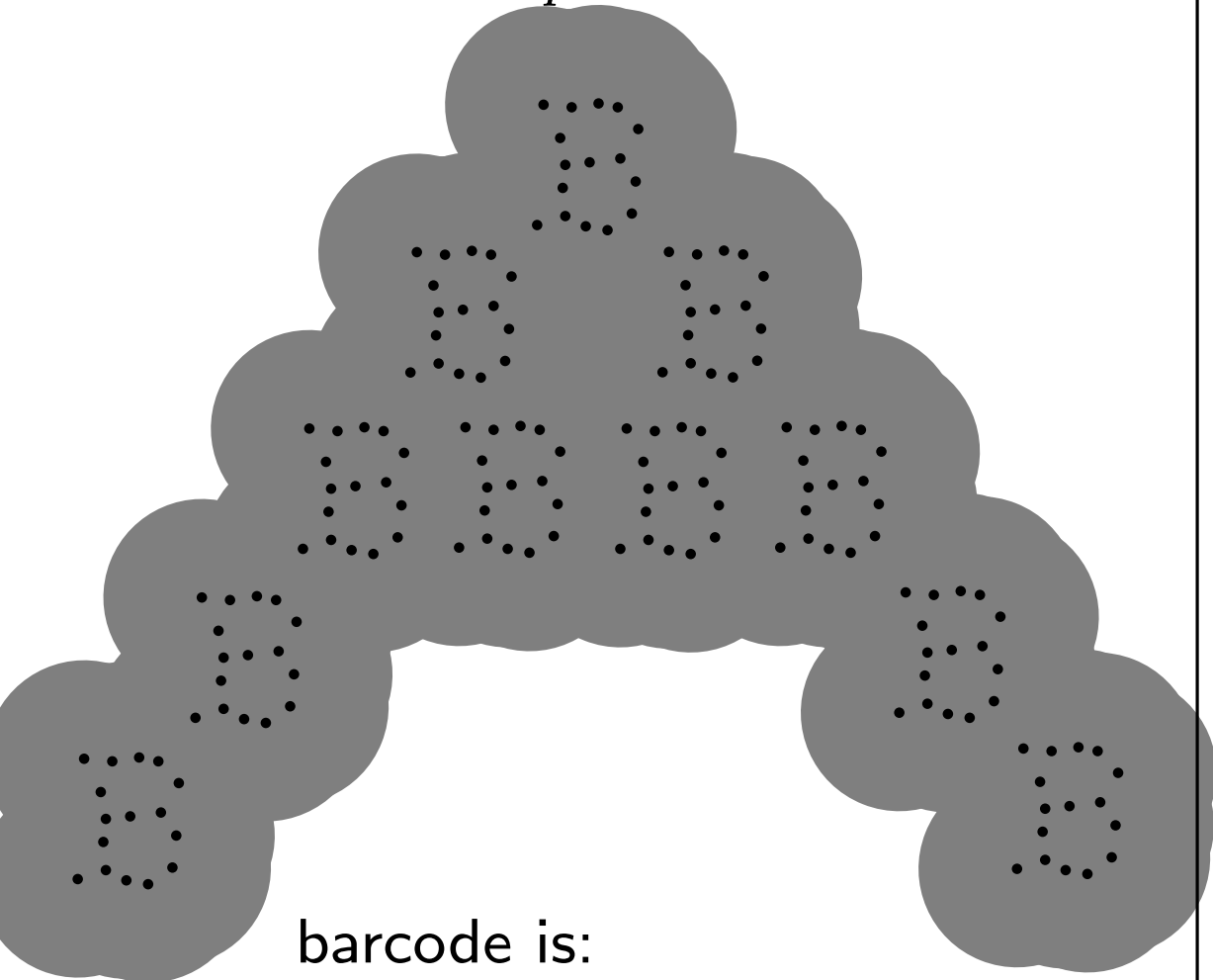
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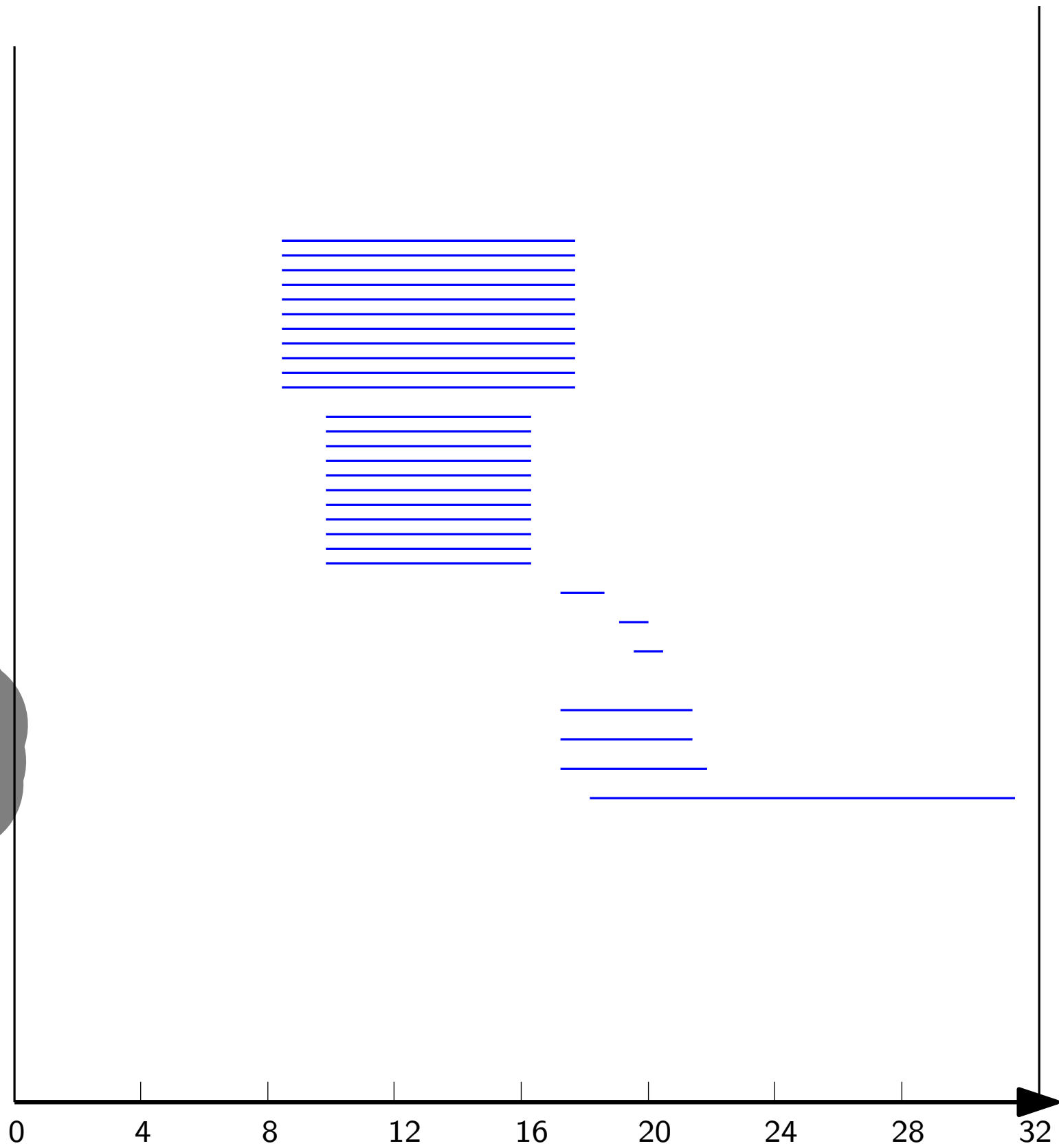
Distance Function

$$d_{\mathcal{P}} : \mathbb{R}^2 \rightarrow \mathbb{R}$$

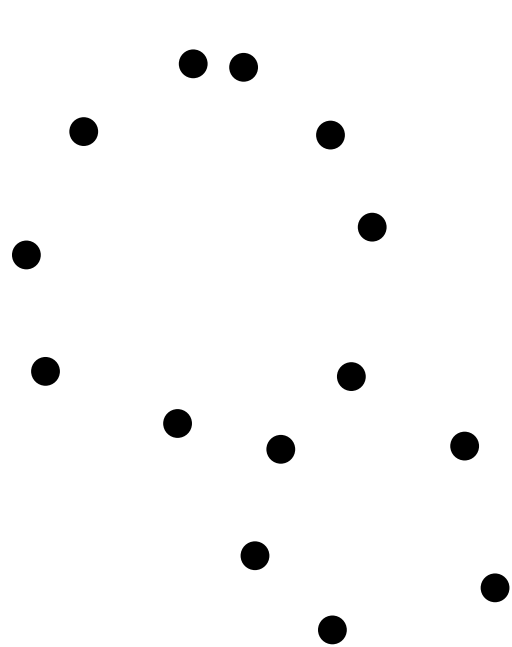
$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$



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- more stable
- generalizable

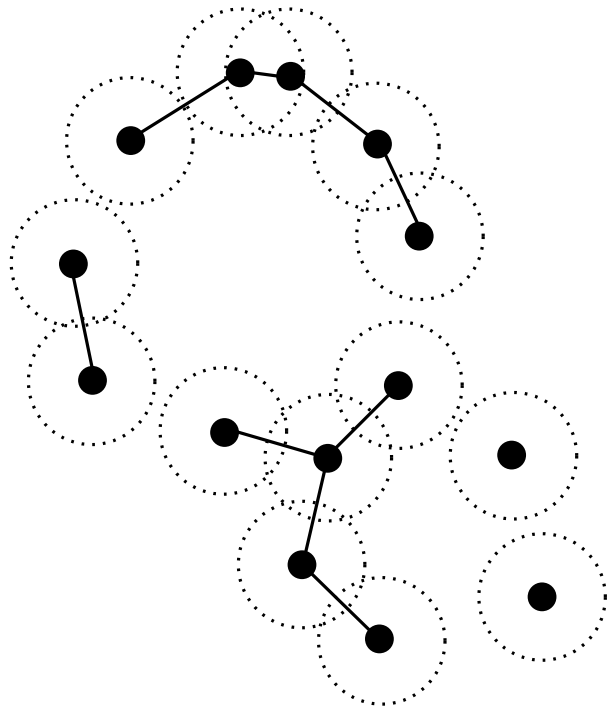


Persistence for Point Cloud Data



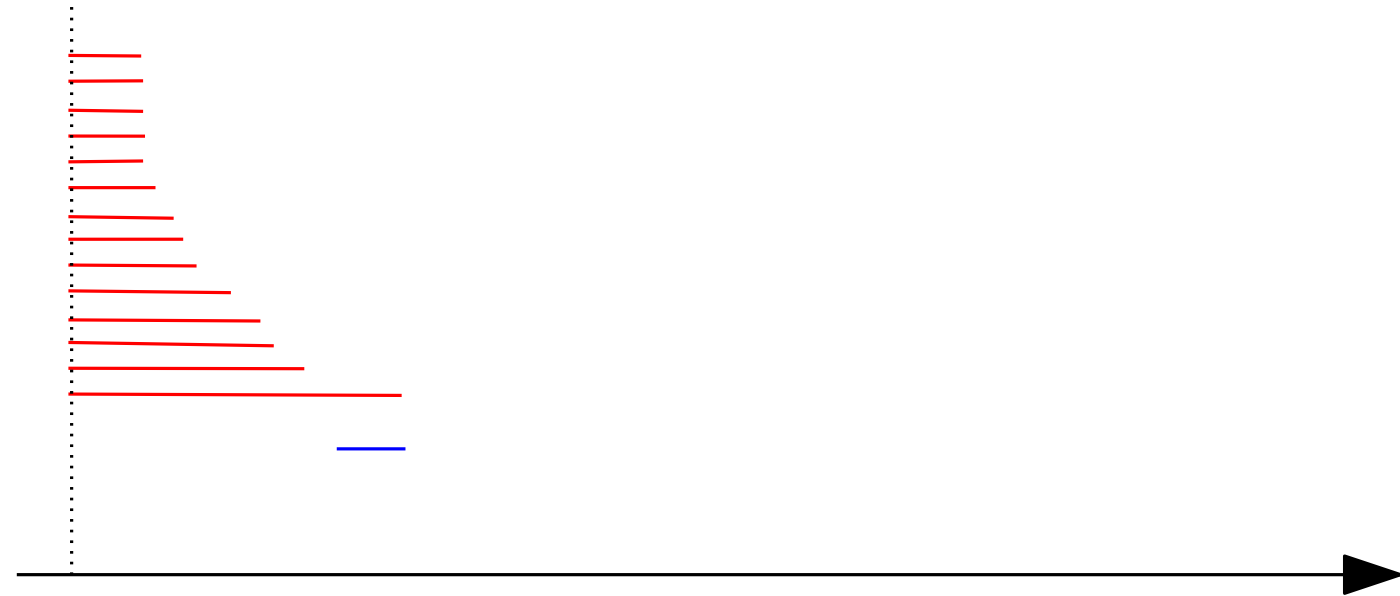
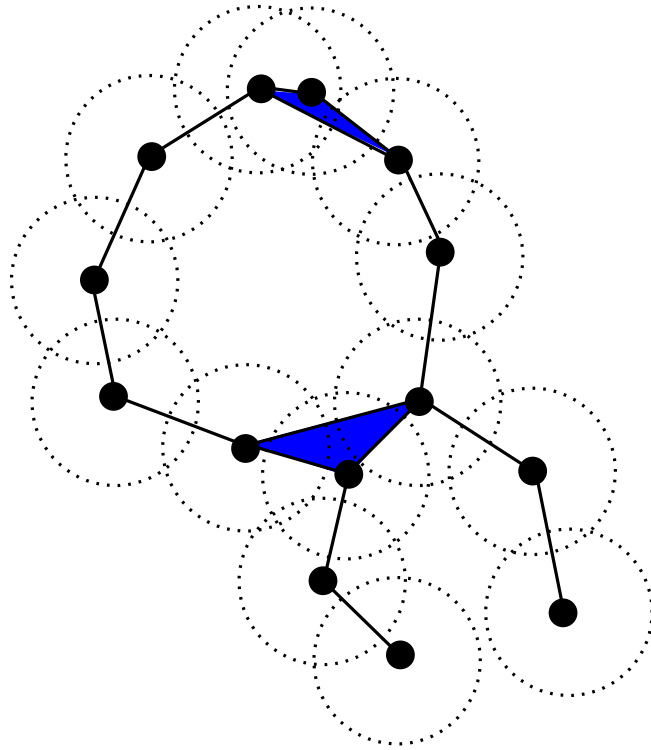
$$\begin{aligned} d_{\mathcal{P}} : \quad & \mathbb{R}^2 \rightarrow \mathbb{R} \\ & x \mapsto \min_{p \in \mathcal{P}} \|x - p\| \end{aligned}$$

Persistence for Point Cloud Data



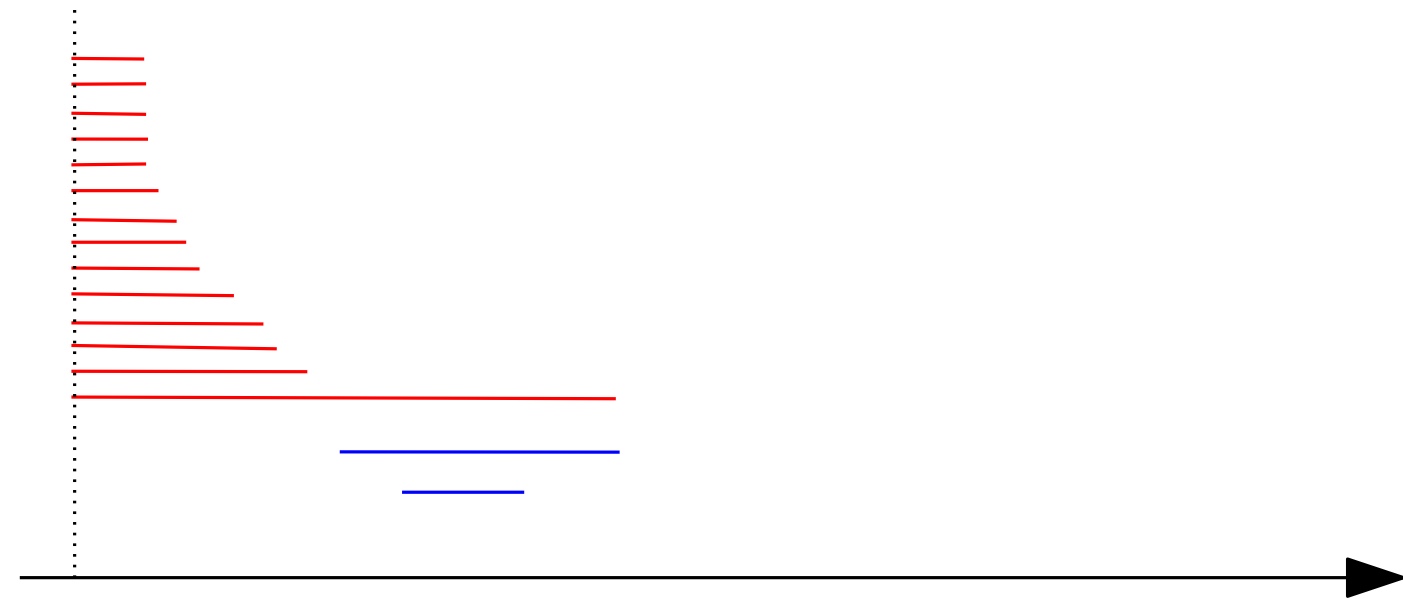
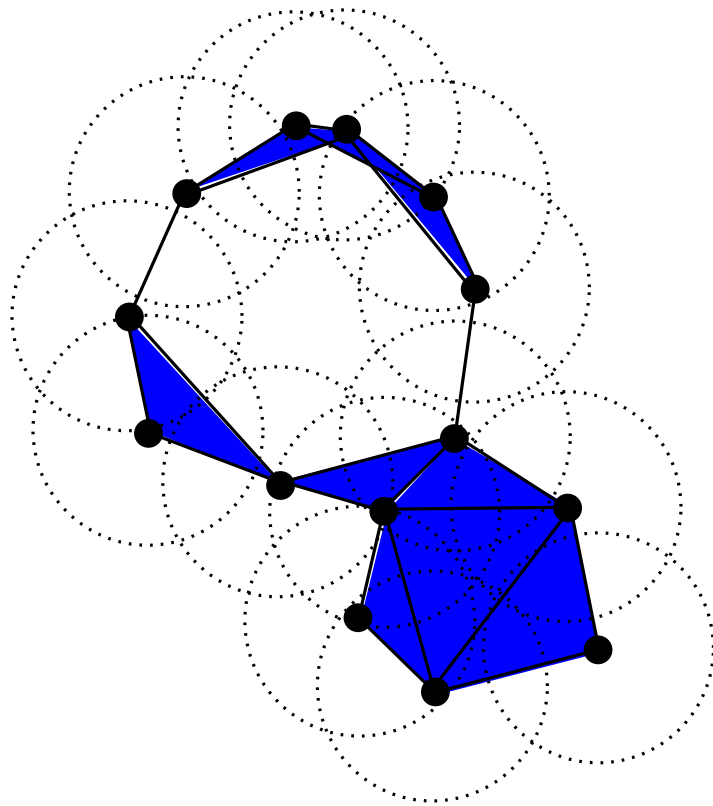
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Persistence for Point Cloud Data



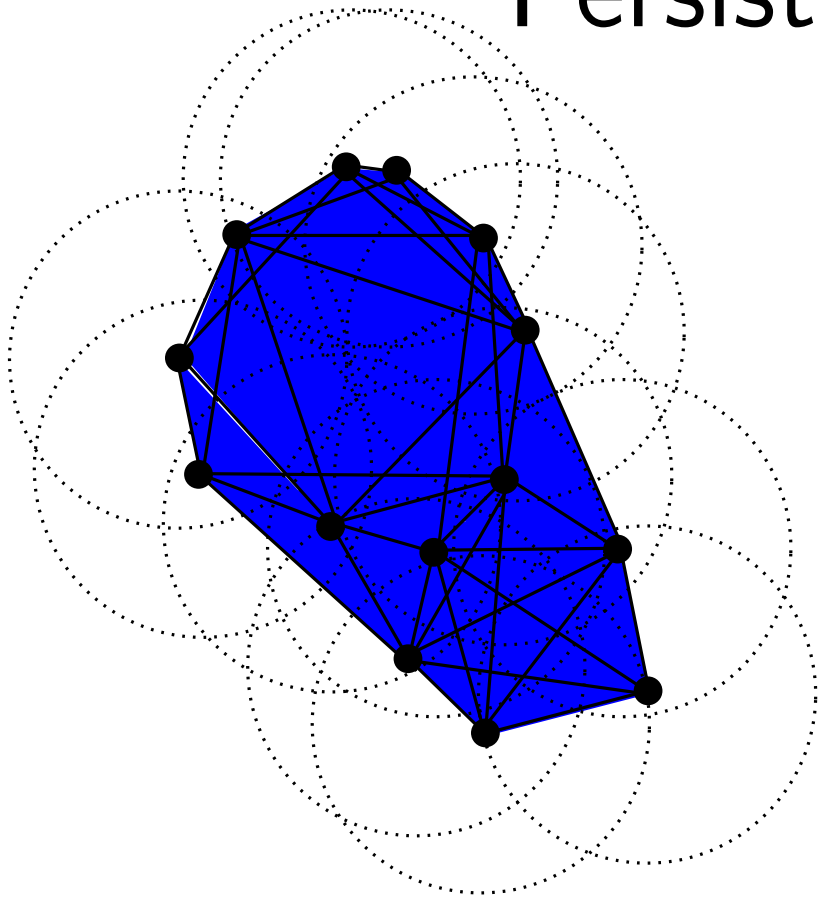
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Persistence for Point Cloud Data

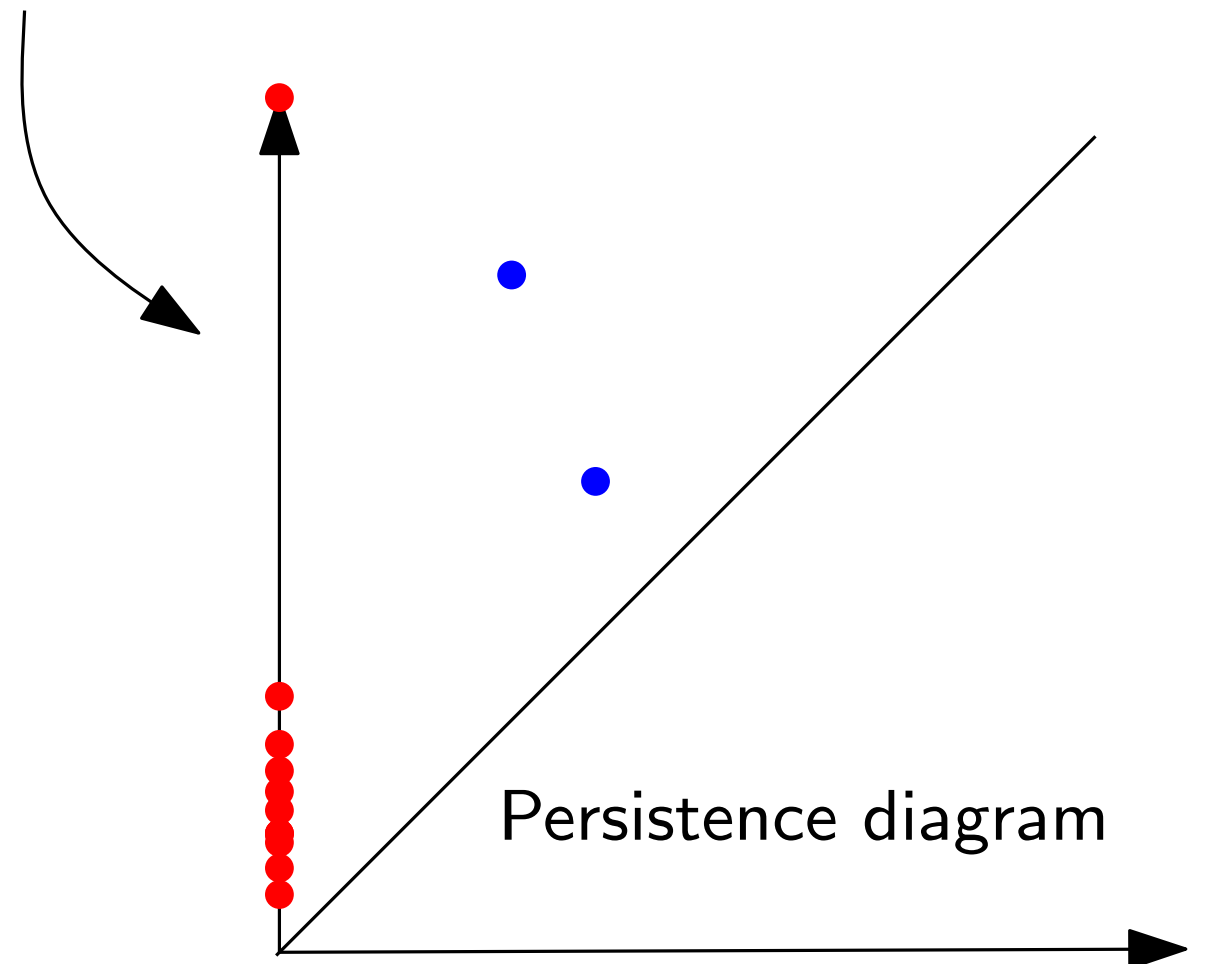
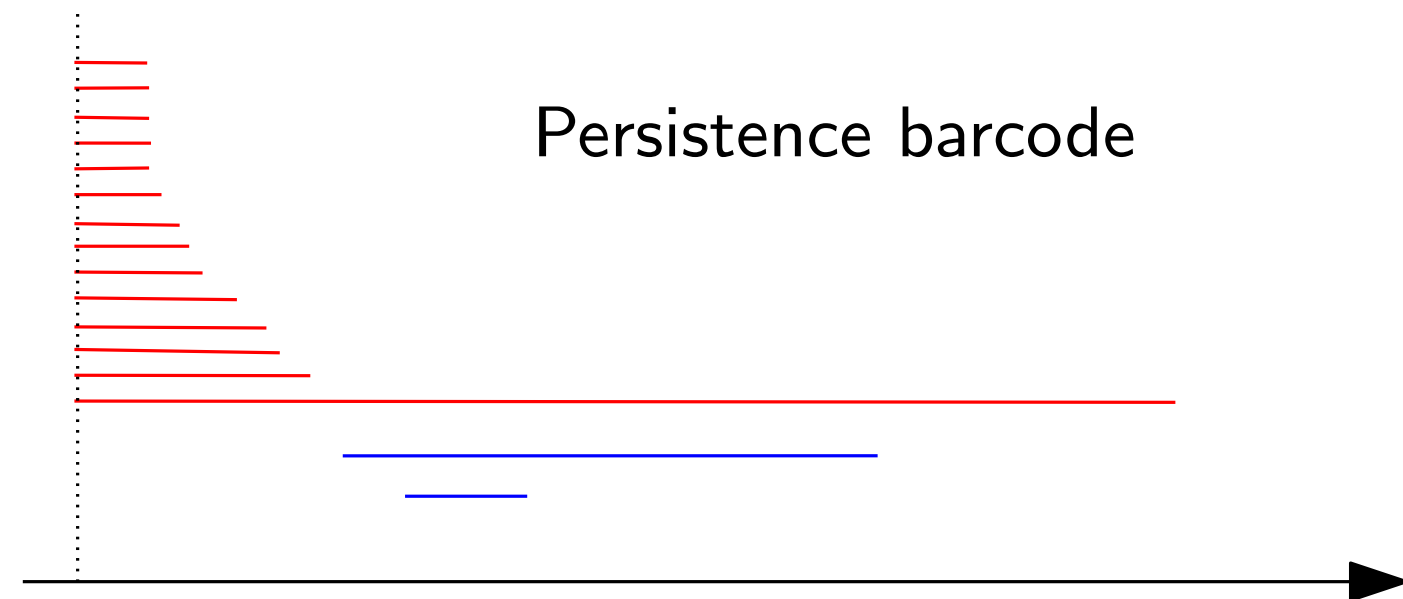


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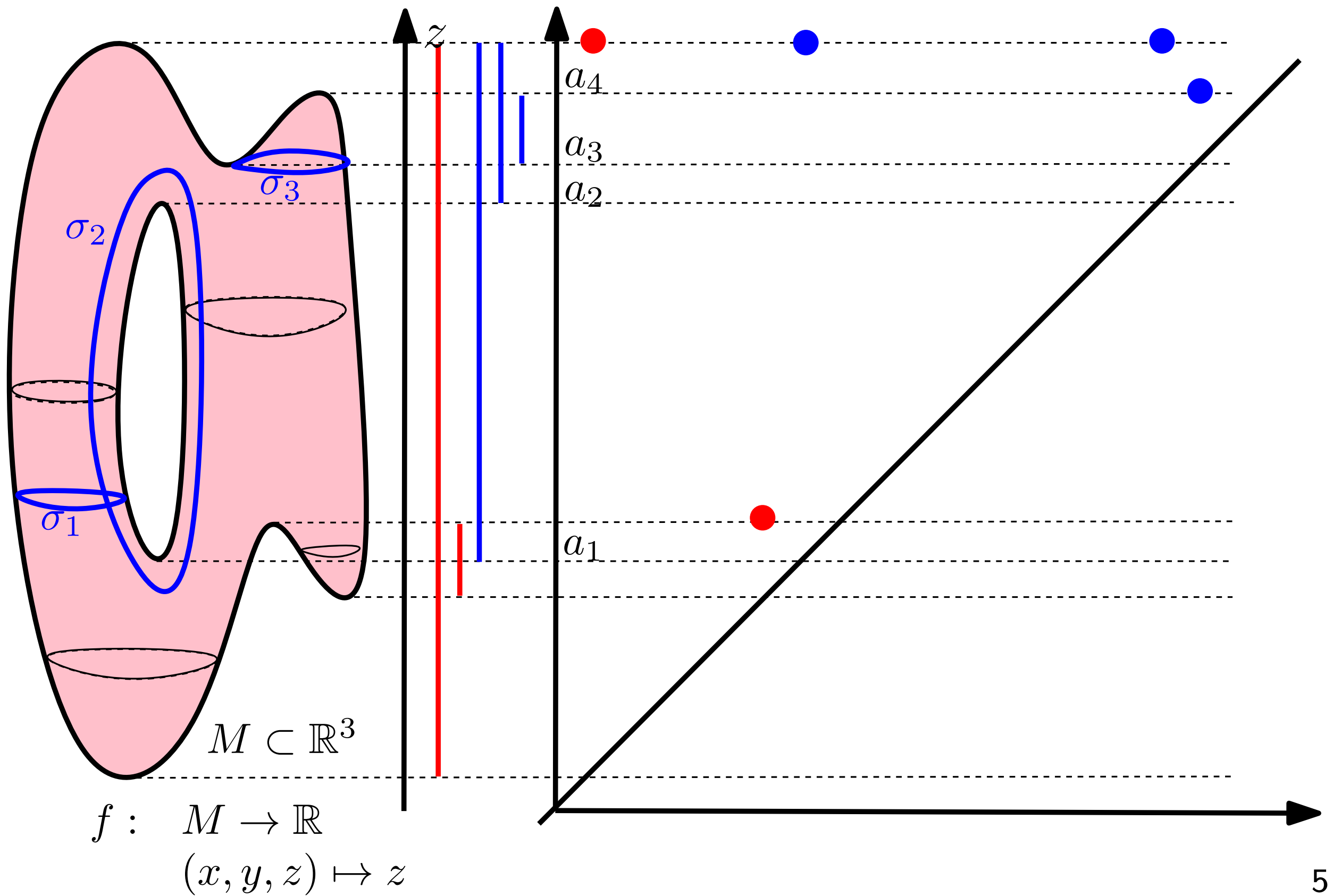
Persistence for Point Cloud Data



$$d_{\mathcal{P}} : \mathbb{R}^2 \rightarrow \mathbb{R}$$
$$x \mapsto \min_{p \in \mathcal{P}} \|x - p\|$$



Surface in Space



Persistence Stability

Theorem: For any “nice” functions $f, g : X \rightarrow \mathbb{R}$,

$$d_b(\text{dgm}_k f, \text{dgm}_k g) \leq \|f - g\|_\infty$$

